HNL see-saw: lower mixing limit and pseudodegenerate state

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10.04.2023

Introduction

Heavy Neutral Leptons (HNL)

$$\mathscr{L} = i\bar{N}_I\gamma^\mu\partial_\mu N_I - \left(\frac{1}{2}M_N\bar{N}_I^c N_I + \hat{Y}_{\alpha I}\bar{L}_\alpha\tilde{H}N_I + h.c.\right) \tag{1}$$

- Renormalizable theory
- SM neutrino mass scale explanation
- Experimental confirmation prospects
- (Optional) Baryon asymmetry mechanism
- (Optional) Dark matter candidate
  Mixing angle:

$$U = \frac{v}{\sqrt{2}}M_R^{-1}Y = iM_R^{-\frac{1}{2}}Rm_\nu^{\frac{1}{2}}U_{PMNS}^\dagger \tag{2}$$

2HNL: parametrization

$$U = i \times \begin{pmatrix} \frac{1}{\sqrt{M_1}}\Gamma_1 & \frac{1}{\sqrt{M_1}}F_1 & \frac{1}{\sqrt{M_1}}G_1 \\ \frac{1}{\sqrt{M_2}}\Gamma_2 & \frac{1}{\sqrt{M_2}}F_2 & \frac{1}{\sqrt{M_2}}G_2 \end{pmatrix}, \tag{3}$$

where

$$\Gamma_1 \equiv \lambda_1 c - \lambda_2 e^{i\Psi} s \tag{4}$$
$$\Gamma_2 \equiv \lambda_1 s + \lambda_2 e^{i\Psi} c \tag{5}$$
$$F_1 \equiv \eta_1 c - \eta_2 e^{i\Psi} s \tag{6}$$
$$F_2 \equiv \eta_1 s + \eta_2 e^{i\Psi} c \tag{7}$$
$$G_1 \equiv \xi_1 A_{23} c - \xi_2 e^{i\Psi} s \tag{8}$$
$$G_2 \equiv \xi_1 s + \xi_2 e^{i\Psi} c \tag{9}$$
$$\omega = x + iy \subset \mathbb{C}, c = \cos\omega, s = \sin\omega, \tag{10}$$
$$A_{ij} \equiv U^{\dagger}_{PMNS_{ij}}|_{\alpha_1=0,\alpha_2=0}, i, j = 1, 2, 3 \tag{11}$$

2HNL: parametrization

Only one independent majorana angle ψ remains.
For normal hierarchy ($m_1 = 0$):

$$\begin{array}{llll} \lambda_1 & \equiv \sqrt{m_2}A_{21}; & \lambda_2 & \equiv \sqrt{m_3}A_{31} \\ \eta_1 & \equiv \sqrt{m_2}A_{22}; & \eta_2 & \equiv \sqrt{m_3}A_{32} \\ \xi_1 & \equiv \sqrt{m_2}A_{23}; & \xi_2 & \equiv \sqrt{m_3}A_{33} \\ e^{i\psi} & = \pm e^{i\frac{-\alpha_2}{2}} & & \end{array} \quad (12)$$

For inverted hierarchy ($m_3 = 0$):

$$\begin{array}{llll} \lambda_1 & \equiv \sqrt{m_1}A_{11}; & \lambda_2 & \equiv \sqrt{m_2}A_{21} \\ \eta_1 & \equiv \sqrt{m_1}A_{12}; & \eta_2 & \equiv \sqrt{m_2}A_{22} \\ \xi_1 & \equiv \sqrt{m_1}A_{13}; & \xi_2 & \equiv \sqrt{m_2}A_{32} \\ e^{i\psi} & = \pm e^{i\frac{\alpha_2-\alpha_1}{2}} & & \end{array} \quad (13)$$

Overall, we have **4 unknown real parameters**: x, y in sterile sector and δ, ψ in active sector.

2HNL: minimal mixing

Firstly, we look at the lower limit of mixing of both HNL with one flavour $U_e^2 = \frac{1}{M_1}|\Gamma_1|^2 + \frac{1}{M_2}|\Gamma_2|^2$. The extremum criterium for ω is:

$$\frac{\partial U_e}{\partial \omega} = -\frac{1}{M_1}\Gamma_1^*\Gamma_2 + \frac{1}{M_2}\Gamma_2^*\Gamma_1 = 0 \tag{14}$$

This equation can have three solutions:

$$\Gamma_1 = 0 \tag{15}$$

$$\Gamma_2 = 0 \tag{16}$$

$$M_1 = M_2, |\Gamma_1| \neq 0, |\Gamma_2| \neq 0, \Gamma_1^*\Gamma_2 - \Gamma_1\Gamma_2^* = 0 \tag{17}$$

All three cases can be furter simlified and described with the same expresion that depends only on δ, ψ.

2HNL: minimal mixing

The resulting expression:

$$U^2_{e_{min}} = \frac{1}{M_{max}} |m_1 A^2_{11} + m_2 A^2_{21} + m_3 A^2_{31}| \equiv \frac{|m_{ee}|}{M_{max}} \tag{18}$$

Here $m_1 = 0$ or $m_3 = 0$ and $M_{max} = max\{M_1, M_2\}$.
Much the same for mixing with other flavours:

$$U^2_{\mu_{min}} = \frac{1}{M_{max}} |m_1 A^2_{12} + m_2 A^2_{22} + m_3 A^2_{32}| \equiv \frac{|m_{\mu\mu}|}{M_{max}} \tag{19}$$

$$U^2_{\tau_{min}} = \frac{1}{M_{max}} |m_1 A^2_{13} + m_2 A^2_{23} + m_3 A^2_{33}| \equiv \frac{|m_{\tau\tau}|}{M_{max}} \tag{20}$$

2HNL: minimal mixing

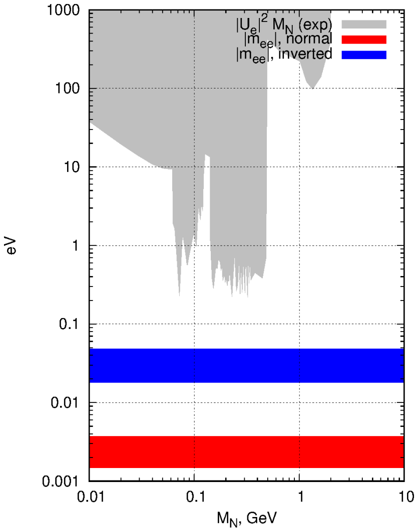


Figure 1: Current experimental limits and possible values of $|m_{ee}|$.

2HNL: pseudodegenerate state

Values $|\Gamma_1|^2$ and $|\Gamma_2|^2$ are closely connected:

$$\begin{aligned} M_1|U_{e1}|^2 + M_2|U_{e2}|^2 &= |\Gamma_1|^2 + |\Gamma_2|^2 \\ &= (|\lambda_1|^2 + |\lambda_2|^2)\cosh(2y) + 2\Im\left[\lambda_1^*\lambda_2 e^{i\Psi}\right]\sinh(2y) \\ &= |m_{ee}|\cosh(2(y-y_e)), \qquad (21) \\ M_1|U_{e1}|^2 - M_2|U_{e2}|^2 &= |\Gamma_1|^2 - |\Gamma_2|^2 \\ &= (|\lambda_1|^2 - |\lambda_2|^2)\cos(2x) - 2\Re\left[\lambda_1^*\lambda_2 e^{i\Psi}\right]\sin(2x) \\ &= |m_{ee}|\cos(2(x-x_e)), \qquad (22) \end{aligned}$$

One can notice that for all x, y:

$$\begin{aligned} |M_1|U_{e1}|^2 - M_2|U_{e2}|^2| &\leq |m_{ee}| \qquad (23) \\ M_1|U_{e1}|^2 + M_2|U_{e2}|^2 &\geq |m_{ee}| \qquad (24) \end{aligned}$$

Similar inequalities can be written for mixing with other two flavours as well.

2HNL: pseudodegenerate state

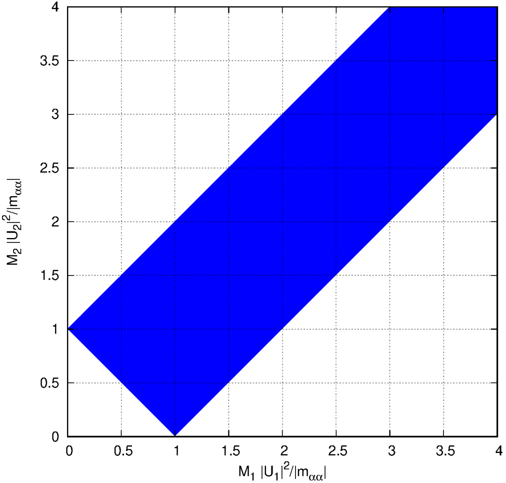


Figure 2: Available region for $M_1|U_{\alpha 1}|^2$ and $M_2|U_{\alpha 2}|^2$, where $\alpha \subset \{e, \mu, \tau\}$

2HNL: pseudodegenerate state

Pseudodegenerate limit:

$$|M_1|U_{e1}|^2 - M_2|U_{e2}|^2| \ll M_1|U_{e1}|^2 + M_2|U_{e2}|^2 \quad (25)$$

That leads to: $M_1|U_{e1}|^2 \approx M_2|U_{e2}|^2$. This limit is automatically achieved for big values of $M|U|^2$:

$$M_1|U_{\alpha 1}|^2 + M_2|U_{\alpha 2}|^2 \gg |m_{ee}| \geq |M_1|U_{\alpha 1}|^2 - M_2|U_{\alpha 2}|^2| \quad (26)$$

In this limit:

$$\sinh(2y) \approx -2M_1|U_{e1}|^2 \frac{2\Im\left[\lambda_1^*\lambda_2 e^{i\psi}\right] \pm (|\lambda_1|^2 + |\lambda_2|^2)}{|m_{ee}|^2} \approx \mp\cosh(2y)$$

2HNL: pseudodegenerate state

Obviously, these expressions are related to $|m_{\alpha\alpha}|$:

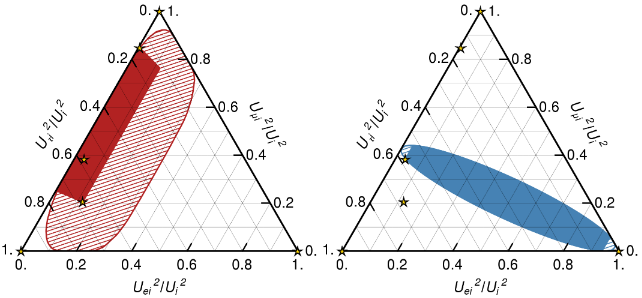
$$|m_{ee}|^2 = \left((|\lambda_1|^2 + |\lambda_2|^2) - 2\Im\left[\lambda_1^* \lambda_2 e^{i\Psi}\right]\right)\left((|\lambda_1|^2 + |\lambda_2|^2) + 2\Im\left[\lambda_1^* \lambda_2 e^{i\Psi}\right]\right), \ldots$$

This allows to find the ratio:

$$\frac{\frac{|U_{ei}|^2}{|U_{\mu i}|^2}}{|U_{\tau i}|^2} = \frac{\frac{\left((|\lambda_1|^2 + |\lambda_2|^2) \mp 2\Im\left[\lambda_1^* \lambda_2 e^{i\Psi}\right]\right)}{\left((|\eta_1|^2 + |\eta_2|^2) \mp 2\Im\left[\eta_1^* \eta_2 e^{i\Psi}\right]\right)}}{\left((|\xi_1|^2 + |\xi_2|^2) \mp 2\Im\left[\xi_1^* \xi_2 e^{i\Psi}\right]\right)} \qquad (27)$$

This ratio is a function of δ and ψ, but doesn't depend on ω or even HNL mass. It allows to draw triangular graphics of $|U_{\alpha,i}|^2/|U_{tot\,i}|^2$ where $|U_{tot\,i}|^2 = |U_{ei}|^2 + |U_{\mu i}|^2 + |U_{\tau i}|^2$.

2HNL: pseudodegenerate state



(a) Normal ordering.

(b) Inverted ordering.

Figure 1: Allowed range of U_a^2/U^2 in the $n = 2$ model for arbitrary parameter choices (hashed region) and in the symmetric limit (filled region) for normal ordering in Panel (a) and inverted ordering in Panel (b) of light neutrino masses. For an experiment with the sensitivity of NA62, the minimal model with $n = 2$ predicts the U_{ai}^2/U_i^2 to lie within the filled areas. Mixing patterns in the extended hashed regions can only be made consistent with light neutrino oscillation data for total $U_i^2 \times M_i/\text{GeV} < 10^{-11}$, cf. Figure 2. The stars mark the benchmark scenarios given in Table 3a.

Figure 3: Fig. 1 from Ref. 1801.04207.

2HNL: pseudodegenerate state

Our results mirror the results obtained in Ref. 1801.04207, both in pictures and in formulae. The crucial difference, though, is that in that work they studied the much stricter "symmetric limit": $M_1 = M_2, |U_{\alpha 1}|^2 = |U_{\alpha 2}|^2$.
We check that the pseudodegenerate limit given by eq. (25) results in the same allowed region as for the limit (26).
The difference is defined by a new parameter
$\kappa_i = \sqrt{1 - \frac{|m_{ee}|^2}{4M_i^2|U_{ei}|^4}}, 0 < \kappa < 1, \kappa_i = 1$ corresponds to the limit (26).

2HNL: pseudodegenerate state

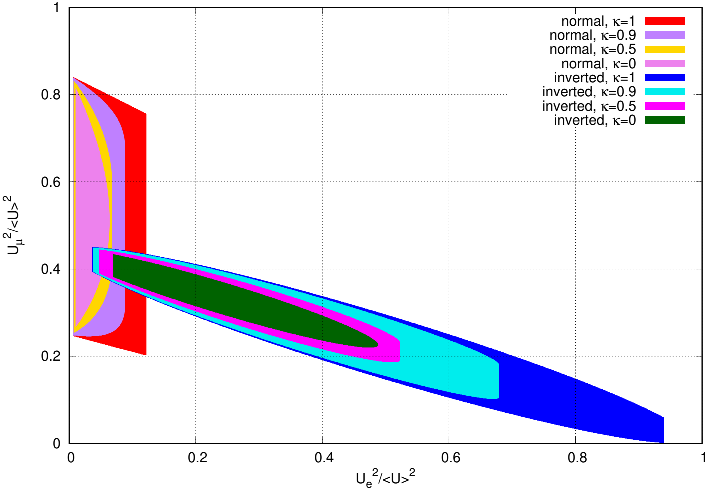


Figure 4: The available regions $(|U_e|^2, |U_\mu|^2)$ in pseudodegenerate limit.

2HNL: pseudodegenerate state

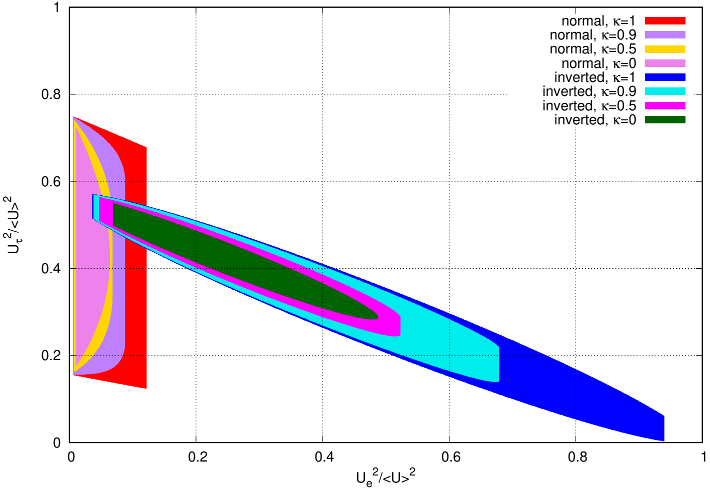


Figure 5: The available regions $(|U_e|^2, |U_\tau|^2)$ in pseudodegenerate limit.

2HNL: pseudodegenerate state

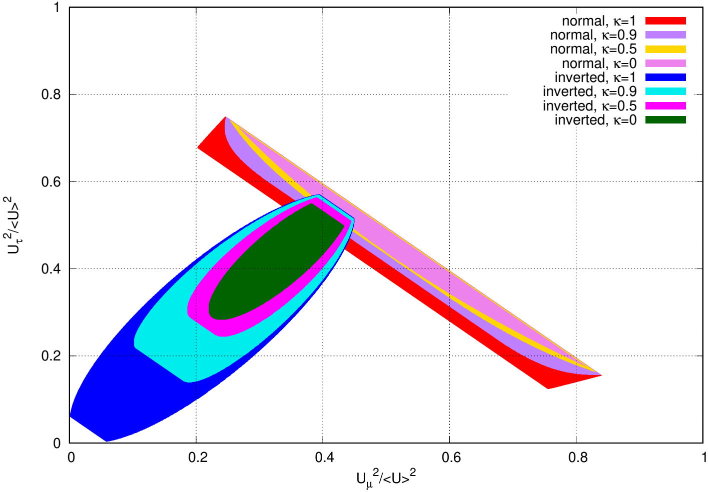


Figure 6: The available regions $(|U_\mu|^2, |U_\tau|^2)$ in pseudodegenerate limit.

2HNL: conclusions

- We have obtained that the minimal value of mixing angle with a given flavour $\alpha \in \{e, \mu, \tau\}$ can be expressed as $\frac{|m_{\alpha\alpha}|}{M_{HNL}}$. That means that if in future experiments for given mass range M_{HNL} we have no evidence of mixing angles higher than the value $\frac{|m_{\alpha\alpha}|}{2M_{HNL}}$, these experiments would rule out see-saw with two HNL in said mass range.
- We state, that, should any evidence of HNL be found in the near future, it would ineveitably mean realization of the pseudodegenerate case (assuming the two HNL scenario is correct). Therefore, the relation to mixing with other flavours would be known to a certain extent and will be the same for both HNLs.

3HNL: parametrization

$$U = i \times diag\{\pm 1, \pm 1, \pm 1\} \times \begin{pmatrix} \frac{1}{\sqrt{M_1}}\Gamma_1 & \frac{1}{\sqrt{M_1}}F_1 & \frac{1}{\sqrt{M_1}}G_1 \\ \frac{1}{\sqrt{M_2}}\Gamma_2 & \frac{1}{\sqrt{M_2}}F_2 & \frac{1}{\sqrt{M_2}}G_2 \\ \frac{1}{\sqrt{M_3}}\Gamma_3 & \frac{1}{\sqrt{M_3}}F_3 & \frac{1}{\sqrt{M_3}}G_3 \end{pmatrix} \tag{28}$$

$$A_{ij} \equiv U^{\dagger}_{PMNS_{ij}}, i, j = 1, 2, 3 \tag{29}$$

$$\lambda_1 \equiv \sqrt{m_1}A_{11}c_1 + \sqrt{m_2}A_{21}s_1, \quad \ldots \tag{30}$$

$$\lambda_2 \equiv \sqrt{m_3}A_{31}, \quad \ldots \tag{31}$$

$$\lambda_3 \equiv -\sqrt{m_1}A_{11}s_1 + \sqrt{m_2}A_{21}c_1, \quad \ldots \tag{32}$$

$$\Gamma_1 \equiv \lambda_1 c_2 + \lambda_2 s_2, \quad \ldots \tag{33}$$

$$\Gamma_4 \equiv \lambda_2 c_2 - \lambda_1 s_2, \quad \ldots \tag{34}$$

$$\Gamma_2 \equiv \lambda_3 c_3 + \Gamma_4 s_3, \quad \ldots \tag{35}$$

$$\Gamma_3 \equiv \Gamma_4 c_3 - \lambda_3 s_3, \quad \ldots \tag{36}$$

Overall, we have **9 unknown real parameters.**

3HNL: minimal mixing

We study:

$$U_e \equiv |U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 = \frac{1}{M_1}|\Gamma_1|^2 + \frac{1}{M_2}|\Gamma_2|^2 + \frac{1}{M_3}|\Gamma_3|^2 \qquad (37)$$

One can find the same limit as in 2HNL case:

$$U^2_{e_{min}} = \frac{1}{M_{max}}|m_1 A^2_{11} + m_2 A^2_{21} + m_3 A^2_{31}| \equiv \frac{|m_{ee}|}{M_{max}} \qquad (38)$$

$$U^2_{\mu_{min}} = \frac{1}{M_{max}}|m_1 A^2_{12} + m_2 A^2_{22} + m_3 A^2_{32}| \equiv \frac{|m_{\mu\mu}|}{M_{max}} \qquad (39)$$

$$U^2_{\tau_{min}} = \frac{1}{M_{max}}|m_1 A^2_{13} + m_2 A^2_{23} + m_3 A^2_{33}| \equiv \frac{|m_{\tau\tau}|}{M_{max}} \qquad (40)$$

3HNL: minimal mixing

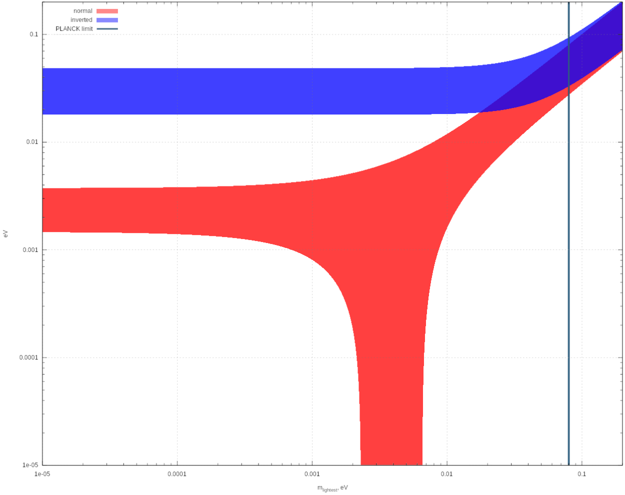


Figure 7: The available values of $|m_{ee}|$ as a function of $m_{lightest}$.

3HNL: minimal mixing

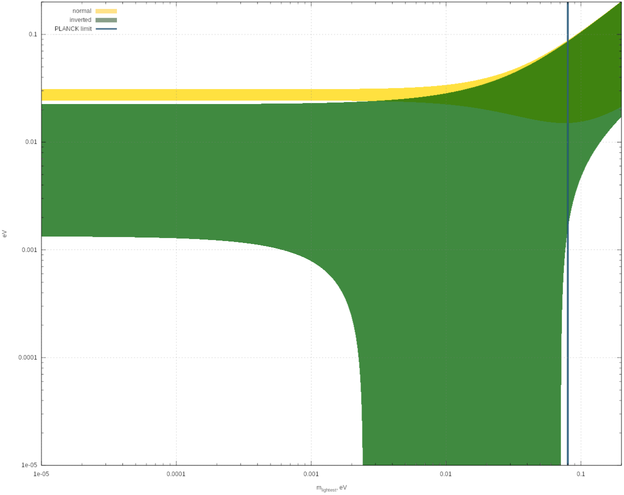


Figure 8: The available values of $|m_{\mu\mu}|$ as a function of $m_{lightest}$.

3HNL: minimal mixing

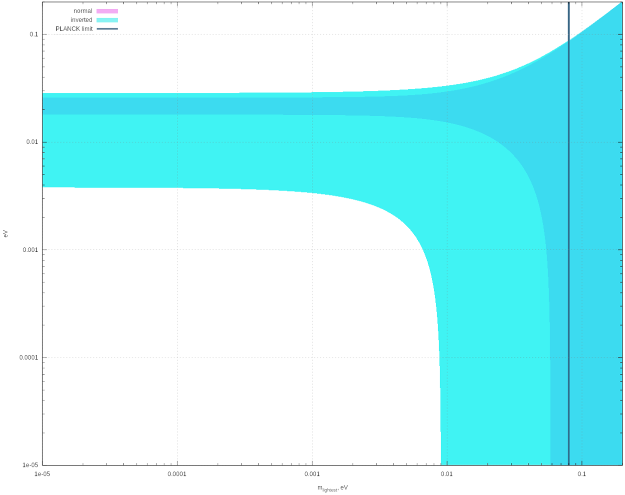


Figure 9: The available values of $|m_{\tau\tau}|$ as a function of $m_{lightest}$.

3HNL: minimal mixing

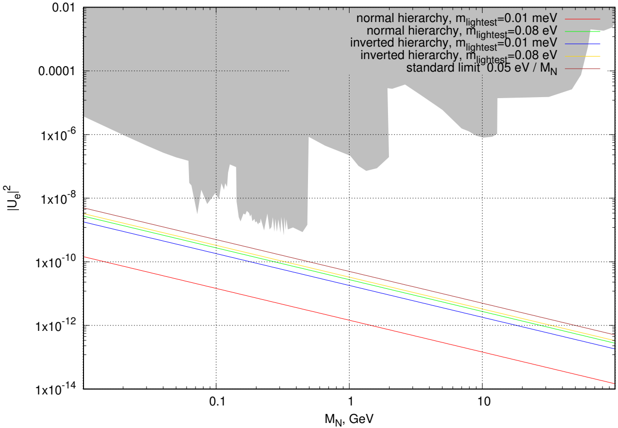


Figure 10: Experimental limits and our see-saw lower boundary for different values of $m_{lightest}$ and usually adopted in literature limit $\frac{0.05\text{eV}}{M_N}$.

3HNL: minimal mixing

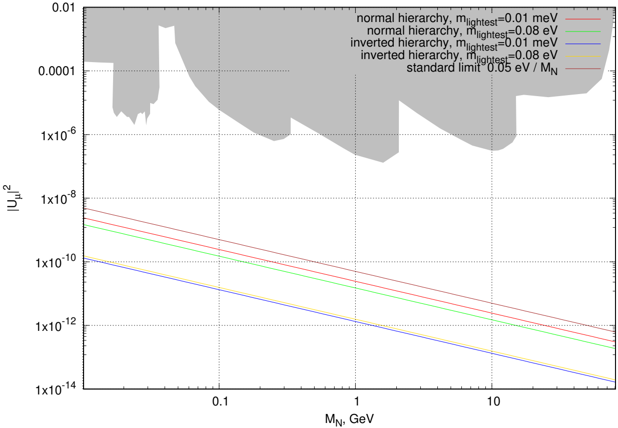


Figure 11: Experimental limits and our see-saw lower boundary for different values of $m_{lightest}$ and usually adopted in literature limit $\frac{0.05\text{eV}}{M_N}$.

3HNL: minimal mixing

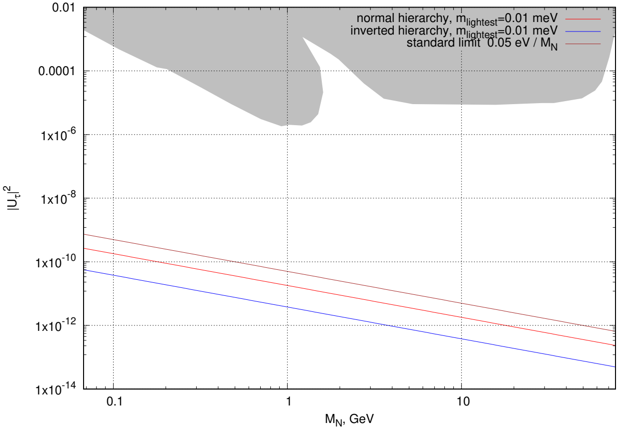


Figure 12: Experimental limits and our see-saw lower boundary for different values of $m_{lightest}$ and usually adopted in literature limit $\frac{0.05\text{eV}}{M_N}$.

3HNL: pseudodegenerate state

One can notice:

$$\left| M_2|U_{e2}|^2 - M_3|U_{e3}|^2 \right| - |m_{ee}| \leq M_1|U_{e1}|^2| \leq M_2|U_{e2}|^2 + M_3|U_{e3}|^2 + |m_{ee}| \quad (41)$$

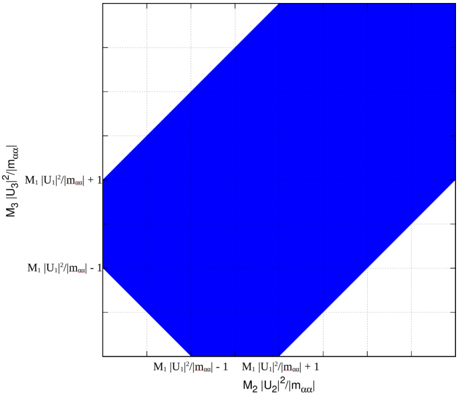


Figure 13: Available region for $M_2|U_{\alpha 2}|^2$ and $M_3|U_{\alpha 3}|^2$, where $\alpha \subset \{e, \mu, \tau\}$.

3HNL: pseudodegenerate state

Pseudodegenerate limit:

$$|M_2|U_{\alpha 2}|^2 - M_3|U_{\alpha 3}|^2| \ll M_2|U_{\alpha 2}|^2 + M_3|U_{\alpha 3}|^2 \tag{42}$$

It is achieved automatically when:

$$M_1|U_{\alpha 1}|^2 \ll M_2|U_{\alpha 2}|^2 + M_3|U_{\alpha 3}|^2 \tag{43}$$

$$|m_{\alpha\alpha}| \ll M_2|U_{\alpha 2}|^2 + M_3|U_{\alpha 3}|^2 \tag{44}$$

One can obtain once again ratio $|U_{ei}|^2 : |U_{\mu i}|^2 : |U_{\tau i}|^2$ as a function of $\delta, \alpha_1, \alpha_2$ and z_1, z_2 that doesn't depend on z_3 and HNL mass M_i.

3HNL: pseudodegenerate state

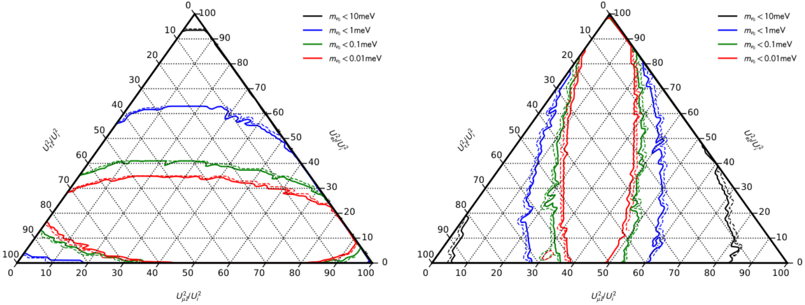


Fig. 11: $U^2_{\alpha I}/U^2_I$ (in percent) for different upper limits of m_{ν_0} (see legend). Solid (dashed) lines delineate the 1σ (2σ) contours, for normal (left) and inverted hierarchy (right). As discussed in footnote 12, these constraints apply to those heavy neutrinos that can be found experimentally.

Figure 14: Ref. 1908.02302.

3HNL: pseudodegenerate state

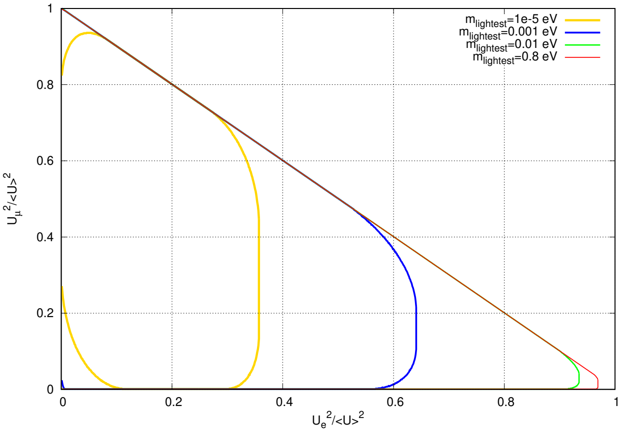


Figure 15: The available regions $(|U_e|^2, |U_\mu|^2)$ in pseudodegenerate limit (normal hirearchy)

3HNL: pseudodegenerate state

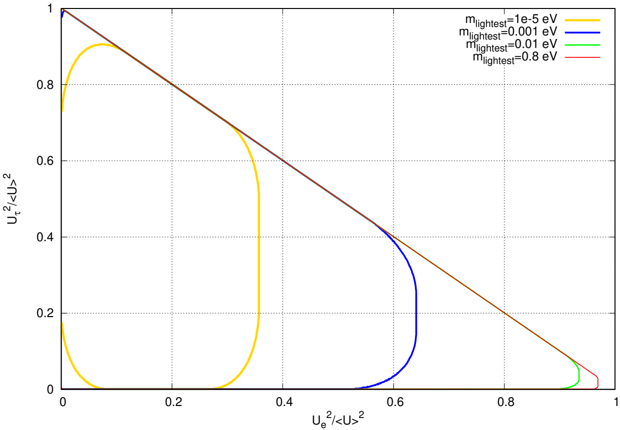


Figure 16: The available regions $(|U_e|^2, |U_\tau|^2)$ in pseudodegenerate limit (normal hirearchy)

3HNL: pseudodegenerate state

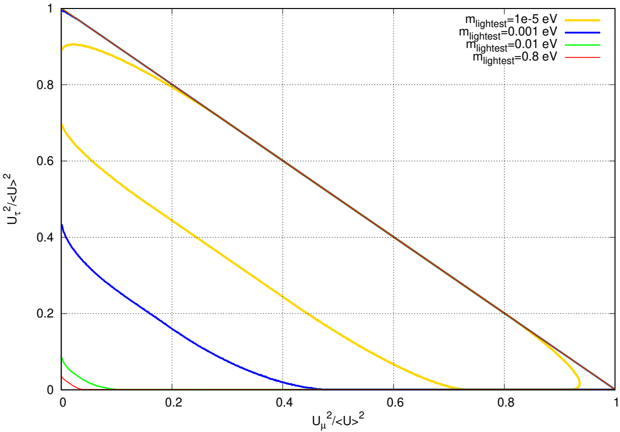


Figure 17: The available regions $(|U_\mu|^2, |U_\tau|^2)$ in pseudodegenerate limit (normal hirearchy)

3HNL: pseudodegenerate state

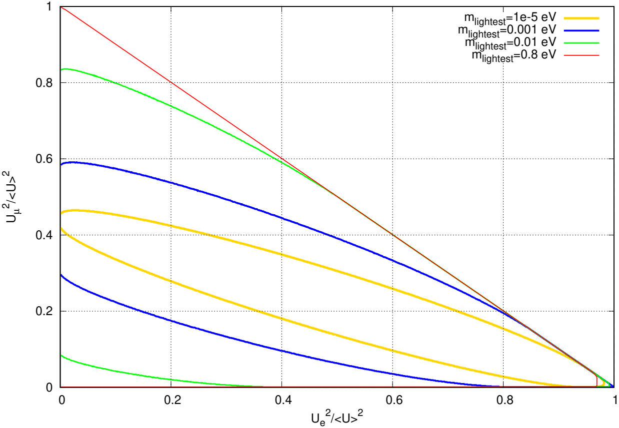


Figure 18: The available regions $(|U_e|^2, |U_\mu|^2)$ in pseudodegenerate limit (inverted hirearchy)

3HNL: pseudodegenerate state

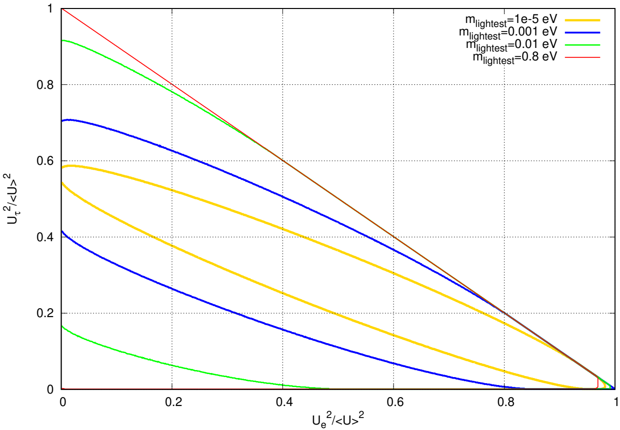


Figure 19: The available regions $(|U_e|^2, |U_\tau|^2)$ in pseudodegenerate limit (inverted hirearchy)

3HNL: pseudodegenerate state

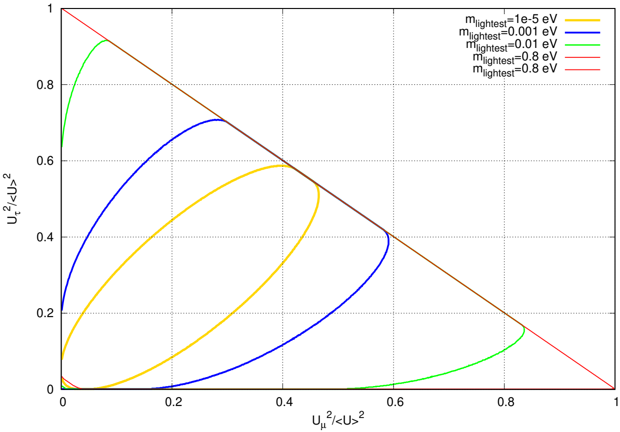


Figure 20: The available regions $(|U_\mu|^2, |U_\tau|^2)$ in pseudodegenerate limit (inverted hirearchy)

3HNL: conclusions

- We have found the minimal line $U_{\alpha i_{min}} = \frac{|m_{\alpha\alpha}|}{M_i}$. $|m_{\alpha\alpha}|$ depend on $m_{lightest}$ and can take zero value. Determination of neutrino mass hierarchy, neutrinoless double beta decay searches and independent restrictions on the sum of neutrino masses can greatly restrict the available region of HNL mixing parameters.
- We have shown that either mixing angle for electron or muon have "solid" boundary $\frac{|m_{\alpha\alpha}|}{3M_{HNL}}$, reaching which would allow to rule out see-saw mechanism in a studied HNL mass region.
- We state that the discovery of HNL signal can provide an insight of where to look for its mixing with other flavours and a hint for the search of another HNL.

Conclusions

- We have found the seesaw lower limit $U_{\alpha i_{min}} = \frac{|m_{\alpha\alpha}|}{M_i}$, both for two- and three-HNL cases, which lay significantly lower than currently adopted see-saw limit.
- We have shown that the new limit can turn to zero for certain values of $m_{lightest}$ and neutrino CP-violating phases, but either $|U_{e_{min}}|^2$ or $|U_{\mu_{min}}|^2$ are guaranteed to be non-zero depending on neutrino mass hierarchy.
- We have found that for areas significantly above this limit the pseudodegenerate limit is achieved automatically. In that case the mixing angles are closely related to each other and fixing one mixing angle's value puts boundaries on the values of other mixing asngles.
- We have shown that results obtained for symmetrical limit, already studied in literature, can be generalized to the pseudo-degenerate limit.

Thank you for attention!

Mass hierarchy

For the normal hierarchy we have:

$$
\begin{aligned}
m_1 &= m_{lightest} \\
m_2 &= \sqrt{m_{lightest}^2 + \Delta m_{21}^2} \\
m_3 &= \sqrt{m_{lightest}^2 + \Delta m_{21}^2 + |\Delta m_{32}^2|},
\end{aligned}
$$

and for the inverted hierarchy:

$$
\begin{aligned}
m_3 &= m_{lightest} \\
m_1 &= \sqrt{m_{lightest}^2 - \Delta m_{21}^2 + |\Delta m_{32}^2|} \\
m_2 &= \sqrt{m_{lightest}^2 + |\Delta m_{32}^2|}.
\end{aligned}
$$

U_{PMNS}

Of utmost importance is mixing angle that describes the relation of flavour basis to mass basis described using Pontecorvo-Maki-Nakagawa-Sakata matrix U_{PMNS}:

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U^{\dagger}_{PMNS} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \tag{45}$$

Here $U^{\dagger}_{PMNS} =$

$$\begin{pmatrix} c_{13}c_{12}e^{i\frac{\alpha_1}{2}} & (-c_{23}s_{12} - s_{23}s_{13}c_{12}e^{-i\delta})e^{i\frac{\alpha_1}{2}} & (s_{23}s_{12} - c_{23}s_{13}c_{12}e^{-i\delta})e^{i\frac{\alpha_1}{2}} \\ c_{13}s_{12}e^{i\frac{\alpha_2}{2}} & (c_{23}c_{12} - s_{23}s_{13}s_{12}e^{-i\delta})e^{i\frac{\alpha_2}{2}} & (-s_{23}c_{12} - c_{23}s_{13}s_{12}e^{-i\delta})e^{i\frac{\alpha_2}{2}} \\ s_{13}e^{i\delta} & s_{23}c_{13} & c_{23}c_{13} \end{pmatrix} \tag{46}$$

where c_{ij} and s_{ij} stand for $\cos\theta_{ij}$ and $\sin\theta_{ij}$, with $i, j = 1, 2, 3, i < j$.

2HNL: minimal mixing

$$\begin{aligned} \Gamma_1^2+\Gamma_2^2 &= \left(\lambda_1 c-\lambda_2 e^{i\psi}s\right)^2+\left(\lambda_1 s+\lambda_2 e^{i\psi}c\right)^2=\lambda_1^2+\lambda_2^2 e^{2i\psi} \\ m_1=0 &\rightarrow \lambda_1^2+\lambda_2^2 e^{2i\psi}=\left(\sqrt{m_2}A_{21}\right)^2+\left(\sqrt{m_3}A_{31}\right)^2 e^{-i\alpha_2} \quad (47) \\ m_3=0 &\rightarrow \lambda_1^2+\lambda_2^2 e^{2i\psi}=\left(\sqrt{m_1}A_{11}\right)^2+\left(\sqrt{m_2}A_{21}\right)^2 e^{i(\alpha_2-\alpha_1)} \quad (48) \\ |\Gamma_1^2+\Gamma_2^2| &= |m_1 A_{11}^2 e^{i\alpha_1}+m_2 A_{21}^2 e^{i\alpha_2}+m_3 A_{31}^2| \quad (49) \end{aligned}$$

It doesn't depend on ω.
The expression $|m_{ee}|=|m_1 A_{11}^2 e^{i\alpha_1}+m_2 A_{21}^2 e^{i\alpha_2}+m_3 A_{31}^2|$ itself is also a known entity that appears as an effective neutrino mass in neutrinoless double beta decay searches.
$\Gamma_1=0 \rightarrow |\Gamma_2|^2=|\Gamma_1^2+\Gamma_2^2|=|m_{ee}|$.
$\Gamma_2=0 \rightarrow |\Gamma_1|^2=|\Gamma_1^2+\Gamma_2^2|=|m_{ee}|$.

If $M_1 = M_2 \equiv M$ the mixing angle is:

$$U_e^2 = \frac{1}{M}\left(|\Gamma_1|^2 + |\Gamma_2|^2\right) = \frac{1}{M}\left(\left(|\lambda_1|^2 + |\lambda_2|^2\right)\cosh(2y) + 2\Im\left[\lambda_1^*\lambda_2 e^{i\Psi}\right]\sinh(2y)\right). \quad (50)$$

U_e doesn't depend on x anymore. The $y = y_e$ extremum condition is:

$$\left(|\lambda_1|^2 + |\lambda_2|^2\right)\sinh(2y_e) + 2\Im\left[\lambda_1^*\lambda_2 e^{i\Psi}\right]\cosh(2y_e) = 0 \quad (51)$$

Mixing angle U_e can be, therefore, written as:

$$\begin{aligned} |U_e|^2 &= \frac{1}{M}\sqrt{(|\lambda_1|^2 + |\lambda_2|^2)^2 - 4(\Im\left[\lambda_1^*\lambda_2 e^{i\Psi}\right])^2}\cosh(2(y - y_e)) \\ &= \frac{|m_{ee}|}{M}\cosh(2(y - y_e)) \end{aligned} \quad (52)$$

We obtain the same expression $|m_{ee}|/M$ and the minimum is achieved at $y = y_e$ giving us the same result.

2HNL: pseudodegenerate limit

Once again:

$$\left|M_1|U_{e1}|^2 - M_2|U_{e2}|^2\right| \ll M_1|U_{e1}|^2 + M_2|U_{e2}|^2 \tag{53}$$

Then $\sinh(2y) =$

$$-\frac{2M_1|U_{e1}|^2 2\Im\left[\lambda_1^*\lambda_2 e^{i\psi}\right] \pm (|\lambda_1|^2+|\lambda_2|^2)\sqrt{(2M_1|U_{e1}|^2)^2 - |m_{ee}|^2}}{|m_{ee}|^2} \tag{54}$$

$\cosh(2y) =$

$$\frac{2M_1|U_{e1}|^2(|\lambda_1|^2+|\lambda_2|^2) \pm 2\Im\left[\lambda_1^*\lambda_2 e^{i\psi}\right]\sqrt{(2M_1|U_{e1}|^2)^2 - |m_{ee}|^2}}{|m_{ee}|^2} \tag{55}$$

3HNL parametrization

$$U = \frac{v}{\sqrt{2}} M_R^{-1} Y = i M_R^{-\frac{1}{2}} R m_\nu^{\frac{1}{2}} U^\dagger_{PMNS} \tag{56}$$

$$A_{ij} \equiv U^\dagger_{PMNS_{ij}}, i,j = 1,2,3 \tag{57}$$

$$\lambda_1 \equiv \sqrt{m_1} A_{11} c_1 + \sqrt{m_2} A_{21} s_1 \tag{58}$$
$$\lambda_2 \equiv \sqrt{m_3} A_{31} \tag{59}$$
$$\lambda_3 \equiv -\sqrt{m_1} A_{11} s_1 + \sqrt{m_2} A_{21} c_1 \tag{60}$$
$$\Gamma_1 \equiv \lambda_1 c_2 + \lambda_2 s_2 \tag{61}$$
$$\Gamma_4 \equiv \lambda_2 c_2 - \lambda_1 s_2 \tag{62}$$
$$\Gamma_2 \equiv \lambda_3 c_3 + \Gamma_4 s_3 \tag{63}$$
$$\Gamma_3 \equiv \Gamma_4 c_3 - \lambda_3 s_3 \tag{64}$$

(65)

$$\eta_1 \equiv \sqrt{m_1}A_{12}c_1 + \sqrt{m_2}A_{22}s_1 \tag{66}$$

$$\eta_2 \equiv \sqrt{m_3}A_{32} \tag{67}$$

$$\eta_3 \equiv -\sqrt{m_1}A_{12}s_1 + \sqrt{m_2}A_{22}c_1 \tag{68}$$

$$F_1 \equiv \eta_1 c_2 + \eta_2 s_2 \tag{69}$$

$$F_4 \equiv \eta_2 c_2 - \eta_1 s_2 \tag{70}$$

$$F_2 \equiv \eta_3 c_3 + F_4 s_3 \tag{71}$$

$$F_3 \equiv F_4 c_3 - \eta_3 s_3 \tag{72}$$

$$\tag{73}$$

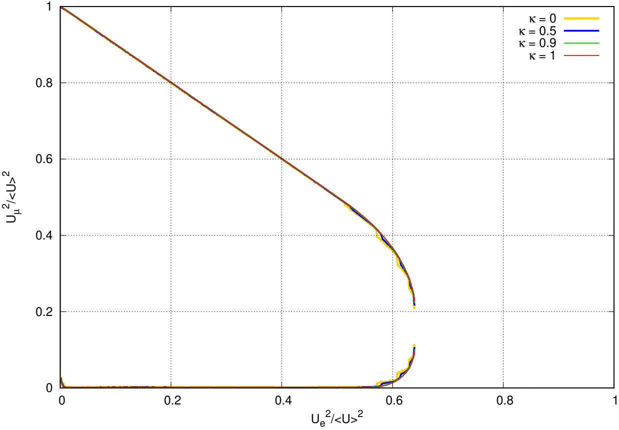
$$
\begin{aligned}
\xi_1 &\equiv \sqrt{m_1}A_{13}c_1 + \sqrt{m_2}A_{23}s_1 && (74)\\
\xi_2 &\equiv \sqrt{m_3}A_{33} && (75)\\
\xi_3 &\equiv -\sqrt{m_1}A_{13}s_1 + \sqrt{m_2}A_{23}c_1 && (76)\\
G_1 &\equiv \xi_1 c_2 + \xi_2 s_2 && (77)\\
G_4 &\equiv \xi_2 c_2 - \xi_1 s_2 && (78)\\
G_2 &\equiv \xi_3 c_3 + G_4 s_3 && (79)\\
G_3 &\equiv G_4 c_3 - \xi_3 s_3 && (80)
\end{aligned}
$$

One can note that $\lambda_1^2+\lambda_3^2=m_1A_{11}^2+m_2A_{21}^2$ does not depend on z_1. In the same way $\Gamma_1^2+\Gamma_4^2=\lambda_1^2+\lambda_2^2$ does not depend on z_2 and $\Gamma_2^2+\Gamma_3^2=\Gamma_4^2+\lambda_3^2$ does not depend on z_3.

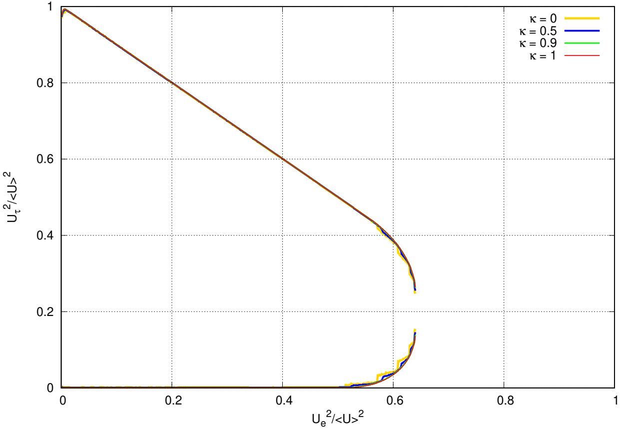
The mixing matrix can be expressed as:

$$U=i\times diag\{\pm 1,\pm 1,\pm 1\}\times\begin{pmatrix} \frac{1}{\sqrt{M_1}}\Gamma_1 & \frac{1}{\sqrt{M_1}}F_1 & \frac{1}{\sqrt{M_1}}G_1 \\ \frac{1}{\sqrt{M_2}}\Gamma_2 & \frac{1}{\sqrt{M_2}}F_2 & \frac{1}{\sqrt{M_2}}G_2 \\ \frac{1}{\sqrt{M_3}}\Gamma_3 & \frac{1}{\sqrt{M_3}}F_3 & \frac{1}{\sqrt{M_3}}G_3 \end{pmatrix} \qquad (81)$$

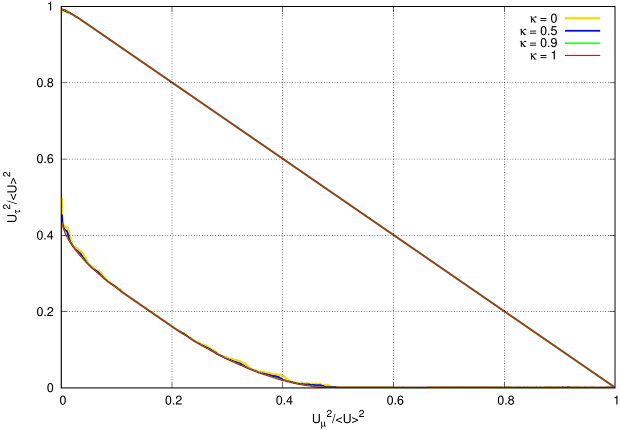
3HNL: pseudodegenerate state



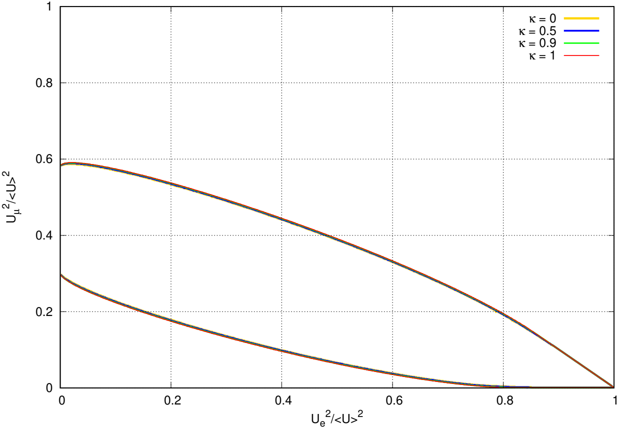
3HNL: pseudodegenerate state



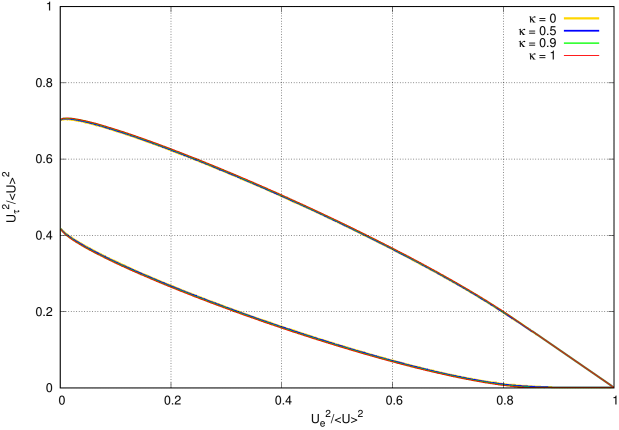
3HNL: pseudodegenerate state



3HNL: pseudodegenerate state



3HNL: pseudodegenerate state



3HNL: pseudodegenerate state