

Parton distribution functions in QED

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Rubakov conference, Yerevan, 2023

2-8 October 2023

Outline

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Motivation

- Development of physical programs for future high-energy e^+e^- colliders
- Having high-precision theoretical description of basic e^+e^- processes is of crucial importance
- Two-loop calculations are in progress, and higher-order QED corrections are also required
- The formalism of QED parton distribution functions can give a fast estimate of the bulk of higher-order effects

Future e^+e^- collider projects

Linear Colliders

- ILC, CLIC
- ILC: technology is ready, not to be built in Japan (?)

E_{tot}

- ILC: 91; 250 GeV — 1 TeV
- CLIC: 500 GeV — 3 TeV

$$\mathcal{L} \approx 2 \cdot 10^{34} \text{ cm}^{-2}\text{s}^{-1}$$

Stat. uncertainty $\sim 10^{-3}$

Circular Colliders

- FCC-ee, TLEP
- CEPC
- $\mu^+\mu^-$ collider (μ TRISTAN)

E_{tot}

- 91; 160; 240; 350 GeV

$$\mathcal{L} \approx 2 \cdot 10^{36} \text{ cm}^{-2}\text{s}^{-1} \text{ (4 exp.)}$$

Stat. uncertainty $\sim 10^{-6}$

Tera-Z mode!

Perturbative QED (I)

Fortunately, in our case the general perturbation theory can be applied:

$$\frac{\alpha}{2\pi} \approx 1.2 \cdot 10^{-3}, \quad \left(\frac{\alpha}{2\pi}\right)^2 \approx 1.4 \cdot 10^{-6}$$

Moreover, other effects: **hadronic vacuum polarization**, **(electro)weak contributions**, **hadronic pair emission**, etc. are small in, e.g., Bhabha scattering and can be treated one-by-one separately

Nevertheless, there are some enhancement factors:

- 1) First of all, the **large logarithm** $L \equiv \ln \frac{\Lambda^2}{m_e^2}$ where $\Lambda^2 \sim Q^2$ is the momentum transferred squared, e.g., $L(\Lambda = 1 \text{ GeV}) \approx 16$ and $L(\Lambda = M_Z) \approx 24$.
- 2) The energy region at the Z boson peak ($s \sim M_Z^2$) requires a special treatment since factor M_Z/Γ_Z appears in the annihilation channel

Perturbative QED (II)

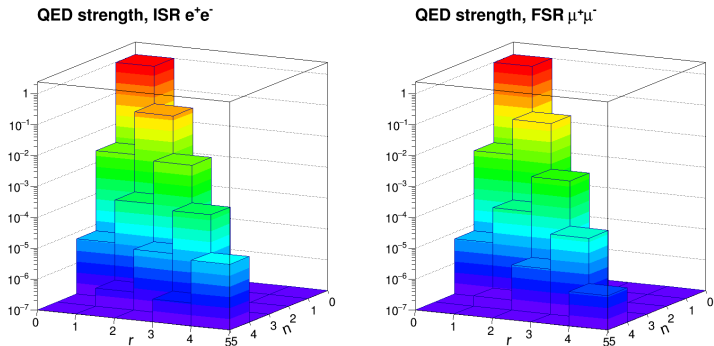


Fig.: The parameter γ_{nr} characterizing the size of the QED corrections,

$$\gamma_{nr} = \left(\frac{\alpha}{\pi}\right)^n \left(2 \ln \frac{M_Z^2}{m_f^2}\right)^r, \quad 1 \leq r \leq n$$

Figure from [S.Jadach and M.Skrzypek, arXiv:1903:09895]

Perturbative QED (III)

Methods of resummation of QED corrections

- Resummation of **vacuum polarization** corrections (geometric series)
- Yennie–Frautschi–Suura (YFS) soft photon exponentiation and its extensions, see, e.g., **PHOTOS**
- Resummation of leading logarithms via **QED structure functions** or **QED PDFs** (E.Kuraev and V.Fadin 1985; A. De Rujula, R. Petronzio, A. Savoy-Navarro 1979)

N.B. Resummation of real photon radiation is good for inclusive observables...

Leading and next-to-leading logs in QED

The QED leading (LO) logarithmic corrections

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^n \frac{s}{m_e^2}$$

were relevant for LEP measurements of Bhabha, $e^+e^- \rightarrow \mu^+\mu^-$ etc.
for $n \leq 3$ since $\ln(M_Z^2/m_e^2) \approx 24$

NLO contributions

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^{n-1} \frac{s}{m_e^2}$$

with $n = 3$ are required for future e^+e^- colliders

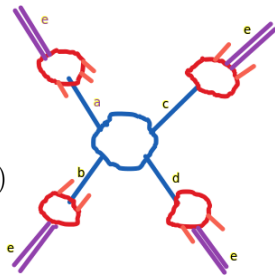
In the collinear approximation we can get them within
the NLO QED structure function formalism

- F.A.Berends, W.L. van Neerven, G.J.Burgers, NPB'1988
- A.A., K.Melnikov, PRD'2002; A.A. JHEP'2003

QED NLO master formula

The **NLO Bhabha** cross section reads

$$\begin{aligned}
 d\sigma &= \sum_{a,b,c,d=e,\bar{e},\gamma} \int_{\bar{z}_1}^1 dz_1 \int_{\bar{z}_2}^1 dz_2 \mathcal{D}_{ae}^{\text{str}}(z_1) \mathcal{D}_{b\bar{e}}^{\text{str}}(z_2) \\
 &\times \left[d\sigma_{ab \rightarrow cd}^{(0)}(z_1, z_2) + d\bar{\sigma}_{ab \rightarrow cd}^{(1)}(z_1, z_2) \right] \\
 &\times \int_{\bar{y}_1}^1 \frac{dy_1}{Y_1} \int_{\bar{y}_2}^1 \frac{dy_2}{Y_2} \mathcal{D}_{ec}^{\text{frg}}\left(\frac{y_1}{Y_1}\right) \mathcal{D}_{\bar{e}d}^{\text{frg}}\left(\frac{y_2}{Y_2}\right) \\
 &+ \mathcal{O}\left(\alpha^n L^{n-2}, \frac{m_e^2}{s}\right)
 \end{aligned}$$



$\alpha^2 L^2$ and $\alpha^2 L^1$ terms are completely reproduced [A.A., E.Scherbakova, JETP Lett. 2006; PLB 2008] || $\bar{e} \equiv e^+$

High-order ISR in e^+e^- annihilation (I)

$$\begin{aligned}
\frac{d\sigma_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s') \left[\mathcal{D}_{e^+e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{e^+e^-} \left(N, \frac{s'}{\mu^2} \right) \mathcal{D}_{e^-e^-} \left(N, \frac{\mu^2}{m_e^2} \right) \right. \\
&+ \mathcal{D}_{\gamma e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{e^-\gamma} \left(N, \frac{s'}{\mu^2} \right) \mathcal{D}_{e^-e^-} \left(N, \frac{\mu^2}{m_e^2} \right) \\
&+ \mathcal{D}_{e^+e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{e^+\gamma} \left(N, \frac{s'}{\mu^2} \right) \mathcal{D}_{\gamma e^-} \left(N, \frac{\mu^2}{m_e^2} \right) \\
&\left. + \mathcal{D}_{\gamma e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{\gamma\gamma} \left(N, \frac{s'}{\mu^2} \right) \mathcal{D}_{\gamma e^-} \left(N, \frac{\mu^2}{m_e^2} \right) \right]
\end{aligned}$$

J. Ablinger, J. Blümlein, A. De Freitas and K. Schönwald,
 “Subleading Logarithmic QED Initial State Corrections to $e^+e^- \rightarrow \gamma^*/Z^{0*}$ to $O(\alpha^6 L^5)$,” NPB 955 (2020) 115045

High-order ISR in e^+e^- annihilation (II)

$$\frac{d\sigma_{e^+e^- \rightarrow \gamma^*}}{ds'} = \frac{1}{s} \sigma^{(0)}(s') \sum_{a,b=e^-, \gamma, e^+} D_{ae^-} \otimes \tilde{\sigma}_{ab \rightarrow \gamma^*} \otimes D_{be^+}$$

Table. Orders of different contributions:

$a \backslash b$	e^+	γ	e^-
e^-	$D_{e^-e^-} D_{e^+e^+} \sigma_{e^-e^+}$ LO (1)	$D_{\gamma e^-} D_{e^-e^-} \sigma_{e^- \gamma}$ NLO ($\alpha^2 L$)	$D_{e^-e^-} D_{e^-e^+} \sigma_{e^-e^-}$ NNLO ($\alpha^4 L^2$)
γ	$D_{\gamma e^-} D_{e^+e^+} \sigma_{e^+ \gamma}$ NLO ($\alpha^2 L$)	$D_{\gamma e^-} D_{\gamma e^+} \sigma_{\gamma \gamma}$ NNLO ($\alpha^4 L^2$)	$D_{\gamma e^-} D_{e^-e^+} \sigma_{e^- \gamma}$ NLO ($\alpha^4 L^3$)
e^+	$D_{e^+e^-} D_{e^+e^+} \sigma_{e^+e^+}$ NNLO ($\alpha^4 L^2$)	$D_{e^+e^-} D_{\gamma e^+} \sigma_{e^+ \gamma}$ NLO ($\alpha^4 L^3$)	$D_{e^+e^-} D_{e^-e^+} \sigma_{e^+e^-}$ LO ($\alpha^4 L^4$)

Contributions from $D_{e^-e^+}$ and $D_{e^+e^-}$ are missed. They are relevant starting from $\mathcal{O}(\alpha^4 L^4)$

QED NLO DGLAP evolution equations

$$D_{ba}\left(x, \frac{\mu_R}{\mu_F}\right) = \delta_{ab}\delta(1-x) + \sum_{c=e,\gamma,\bar{e}} \int_{\mu_R^2}^{\mu_F^2} \frac{dt}{t} \int_x^1 \frac{dy}{y} P_{bc}(y,t) D_{ca}\left(\frac{x}{y}, \frac{\mu_R}{t}\right)$$

μ_F is a **factorization** (energy) scale

μ_R is a **renormalization** (energy) scale

D_{ba} is a parton distribution function (**PDF**)

P_{bc} is a **splitting function** or kernel of the DGLAP equation

N.B. In QED $\mu_R = m_e \approx 0$ is the natural choice

Initial conditions

$\mathcal{D}_{ba}^{\text{ini}}$ is the initial approximation in iterative solutions

$$\mathcal{D}_{ee}^{\text{ini}}(x, \mu_R, m_e) = \delta(1-x) + \frac{\bar{\alpha}(\mu_R)}{2\pi} d_{ee}^{(1)}(x, \mu_R, m_e) + \mathcal{O}(\alpha^2)$$

$$\mathcal{D}_{\gamma e}^{\text{ini}}(x, \mu_R, m_e) = \frac{\bar{\alpha}(\mu_R)}{2\pi} P_{\gamma e}^{(0)}(x) + \mathcal{O}(\alpha^2)$$

$$d_{ee}^{(1)}(x, \mu_R, m_e) = \left[\frac{1+x^2}{1-x} \left(\ln \frac{\mu_R^2}{m_e^2} - 2 \ln(1-x) - 1 \right) \right]_+$$

They are defined from **matching** to perturbative calculations, see below

QED splitting functions

The perturbative splitting functions are

$$P_{ba}(x, \bar{\alpha}(t)) = \frac{\bar{\alpha}(t)}{2\pi} P_{ba}^{(0)}(x) + \left(\frac{\bar{\alpha}(t)}{2\pi} \right)^2 P_{ba}^{(1)}(x) + \mathcal{O}(\alpha^3)$$

$$\text{e.g. } P_{ee}^{(0)}(x) = \left[\frac{1+x^2}{1-x} \right]_+$$

They come from direct loop calculations, see, e.g., review “Partons in QCD” by G. Altarelli. For instance, $P_{ba}^{(1)}(z)$ comes from 2-loop calculations.

The splitting functions can be obtained by reduction of the ones known in QCD to the abelian case of QED.

$\bar{\alpha}(t)$ is the QED running coupling constant in the $\overline{\text{MS}}$ scheme

$\mathcal{O}(\alpha)$ matching

The expansion of the master formula for ISR gives

$$d\sigma_{e\bar{e}\rightarrow\gamma^*}^{(1)} = \frac{\alpha}{2\pi} \left\{ 2LP^{(0)} \otimes d\sigma_{e\bar{e}\rightarrow\gamma^*}^{(0)} + 2d_{ee}^{(1)} \otimes d\sigma_{e\bar{e}\rightarrow\gamma^*}^{(0)} \right\} + d\bar{\sigma}_{e\bar{e}\rightarrow\gamma^*}^{(1)} + \mathcal{O}(\alpha^2)$$

We know the **massive** $d\sigma^{(1)}$ and **massless** $d\bar{\sigma}^{(1)}$ ($m_e \rightarrow 0$ with $\overline{\text{MS}}$ subtraction) results in $\mathcal{O}(\alpha)$. E.g.

$$\frac{d\sigma_{e\bar{e}\rightarrow\gamma^*}^{(1)}}{d\sigma_{e\bar{e}\rightarrow\gamma^*}^{(0)}} = \frac{\alpha}{\pi} \left[\frac{1+z^2}{1-z} \right]_+ \left(\ln \frac{s}{m_e^2} - 1 \right) + \delta(1-z)(\dots), \quad z \equiv \frac{s'}{s}$$

A **scheme dependence** comes from here

A **factorization scale dependence** is also from here

Running coupling constant (I)

Running of α_{QED} is known, e.g., P.Baikov, K.Chetyrkin et al., NPB '2013

$$\bar{\alpha}(t) = \frac{\alpha(\mu_R^2)}{1 + \Pi(\mu_R^2/t)}, \quad \alpha \equiv \alpha(\mu_R^2 = m_e^2) \approx \frac{1}{137.036}$$

$$\implies \bar{\alpha}(t) = \alpha \left\{ 1 + \frac{\alpha}{2\pi} \left(-\frac{10}{9} + \frac{2}{3}L \right) + \left(\frac{\alpha}{2\pi} \right)^2 \left(-\frac{1085}{324} \right. \right.$$

$$\left. \left. + 4\zeta_3 - \frac{13}{27}L + \frac{4}{9}L^2 \right) + \mathcal{O}(\alpha^3(\mu_R)) \right\}$$

The same $\overline{\text{MS}}$ scheme is used here, $L \equiv \ln(t/\mu_R^2)$

Note that here only electron loops are taken into account. Other contributions can be added.

Running coupling constant (II)

Compare **QED-like**

$$\bar{\alpha}(t) = \alpha \left\{ 1 + \frac{\alpha}{2\pi} \left(-\frac{10}{9} + \frac{2}{3}L \right) + \left(\frac{\alpha}{2\pi} \right)^2 \left(-\frac{13}{27}L + \frac{4}{9}L^2 + \dots \right) + \dots \right.$$

and **QCD-like**

$$\bar{\alpha}(t) = \frac{4\pi}{\beta_0 \ln(t/\Lambda^2)} \left[1 - \frac{\beta_1}{\beta_0^2} \frac{\ln[\ln(t/\Lambda^2)]}{\ln(t/\Lambda^2)} + \dots \right]$$

Note that **1** and **-10/9** are hidden into **Λ**

In QED $\beta_0 = -4/3$ and $\beta_1 = -4$

Iterative solution

The **NLO** “electron in electron” PDF reads

$$\begin{aligned}
 \mathcal{D}_{ee}(x, \mu_F, m_e) = & \delta(1-x) + \frac{\alpha}{2\pi} LP_{ee}^{(0)}(x) + \frac{\alpha}{2\pi} d_{ee}^{(1)}(x, m_e, m_e) \\
 & + \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x) + \frac{1}{2} P_{ee}^{(0)}(x) + \frac{1}{2} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}(x) \right) \\
 & + \left(\frac{\alpha}{2\pi}\right)^2 L \left(P_{e\gamma}^{(0)} \otimes d_{\gamma e}^{(1)}(x, m_e, m_e) + P_{ee}^{(0)} \otimes d_{ee}^{(1)}(x, m_e, m_e) + P_{ee}^{(1)}(x) - \frac{10}{9} P_{ee}^{(0)}(x) \right) \\
 & + \left(\frac{\alpha}{2\pi}\right)^3 L^3 \left(\frac{1}{6} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x) + \frac{1}{6} P_{e\gamma}^{(0)} \otimes P_{\gamma\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}(x) + \dots \right) \\
 & + \left(\frac{\alpha}{2\pi}\right)^3 L^2 \left(+\frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes d_{ee}^{(1)}(x, m_e, m_e) + \frac{2}{3} P_{ee}^{(1)}(x) + P_{ee}^{(0)} \otimes P_{ee}^{(1)}(x) - \frac{10}{9} P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x) + \dots \right) \\
 & + \mathcal{O}(\alpha^2 L^0, \alpha^3 L^1)
 \end{aligned}$$

The large logarithm $L \equiv \ln \frac{\mu_F^2}{\mu_R^2}$ with factorization scale $\mu_F^2 \sim s$ or $\sim -t$; and renormalization scale $\mu_R = m_e$.

Required convolution integrals are listed in [A.A. hep-ph/0304063]

Matching in $\mathcal{O}(\alpha^2)$

Complete 2-loop result: Berends et al. 1988; Blümlein et al., 2011

$$\begin{aligned}
 \sigma_{e\bar{e}}^{(2)} = & \left(\frac{\alpha}{2\pi}\right)^2 L^2 \sigma_{e\bar{e}}^{(0)} \left(P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{2}{3} P_{ee}^{(0)} + 2P_{ee}^{(0)} \otimes P_{ee}^{(0)} \right) \\
 & + \left(\frac{\alpha}{2\pi}\right)^2 L \sigma_{e\bar{e}}^{(0)} \left(2d_{\gamma e}^{(1)} \otimes P_{e\gamma}^{(0)} + 2P_{ee}^{(1)} - \frac{40}{9} P_{ee}^{(0)} + 4P_{ee}^{(0)} \otimes d_{ee}^{(1)} \right) \\
 & + \left(\frac{\alpha}{2\pi}\right)^2 L \left(2\sigma_{e\gamma}^{(0)} P_{\gamma e}^{(0)} + \frac{2}{3} \bar{\sigma}_{e\bar{e}}^{(1)} + 2\sigma_{e\bar{e}}^{(1)} \otimes P_{ee}^{(0)} \right) \\
 & + \mathcal{O}(\alpha^2 L^0) + \mathcal{O}(m_e^2/s)
 \end{aligned}$$

Massification procedure

Two-loop (virtual) corrections to Bhabha scattering with $m_e \equiv 0$
[Z.Bern, L.J.Dixon, A.Ghinculov, PRD 2001]

Two-loop (virtual+soft) corrections to Bhabha scattering with $m_e \neq 0$
[A.Penin, PRL 2005; NPB 2006] but for $s, |t|, |u| \gg m_e^2$

Statement: all terms enhanced by large logs $L = \ln(Q^2/m_e^2)$ can be restored

The result of A.Penin was reproduced by adding **universal terms** to the massless result [T.Becher, K.Melnikov, JHEP 2007]

Recently, V. Fadin and R. Lee did the same for $e^+e^- \rightarrow \gamma^*\gamma$
[arXiv:2308.09479]

McMule – NNLO QED Corrections for Low-Energy Experiments
[P.Banerjee, T.Engel, A.Signer, Y.Ulrich, SciPost Phys. 2020]

Applications

Current work:

- ISR in electron-positron annihilation $e^+e^- \rightarrow \gamma^*, Z^*$
“Higher-order NLO initial state radiative corrections to e^+e^- annihilation revisited”
- $\mathcal{O}(\alpha^3 L^2)$ corrections to **muon decay spectrum**: relevant for future experiments on Dirac vs. Majorana neutrino discrimination

Near future plans:

- Implementation into **ZFITTER**, production of benchmarks, tuned comparisons with **KKMC** which uses YFS exponentiation for ISR
- Application to different e^+e^- annihilation channels and asymmetries within the **SANC project**
- $\mathcal{O}(\alpha^3 L^2)$ corrections to muon-electron scattering for **MUonE** experiment

QED PDFs vs. QCD ones

Common properties:

- QED splitting functions = abelian part of QCD ones
- The same structure of DGLAP evolution equations
- The same Drell-Yan-like master formula with factorization
- Factorization scale and scheme dependence

Peculiar properties:

- QED PDFs are calculable
- QED PDFs are less inclusive
- QED renormalization scale $\mu_R = m_e$ is preferable
- QED PDFs can (do) lead to huge corrections
- QED cross-checks QCD

Outlook

- QED NLO PDFs are derived in a consistent way
- Having high theoretical precision for the normalization processes $e^+e^- \rightarrow e^+e^-$, $e^+e^- \rightarrow \mu^+\mu^-$, and $e^+e^- \rightarrow 2\gamma$ is crucial for future e^+e^- colliders, especially for the **Tera-Z mode**
- We need more two-loop QED results for differential distributions. **Massification** procedure helps here
- New **Monte Carlo** codes are required
- Semi-analytic codes are relevant for **cross-checks** and **benchmarks**
- **Comparisons** with recent results of Blümlein et al. show a serious disagreement (even in the leading logs) due to four separate issues
- A **bug** in QCD NLO PDFs is found (?)



”The electron is as inexhaustible as the atom” [V. Lenin '1908]