Observational constrained $f(R, \mathcal{G})$ gravity cosmological model and the dynamical system analysis

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Rubakov Conference – 2023
A. Alikhanyan National Laboratory, Yerevan, Armenia

04.10.2023

Outlines

- Mathematical formalism of $f(R, \mathcal{G})$ gravity
- Observational Constraints
- Stability Analysis of the Model
- Results and Discussion

Mathematical formalism of $f(R,\mathcal{G})$ gravity

The most general action for $f(R,\mathcal{G})$ gravity[1,2]

$$S = \int \sqrt{-g}\left[\frac{1}{2k^2} f(R,\mathcal{G}) + \mathcal{L}_m\right] d^4x \tag{1}$$

The Gauss-Bonnet curvature term is defined as: $\mathcal{G} \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$

By varying the action (1) with respect to the metric tensor g_{ij}, the field equations of the $f(R,\mathcal{G})$ gravity can be expressed as:

$$\begin{aligned} f_R G_{ij} &= k^2 T_{ij} + \frac{1}{2} g_{ij}[f(R,\mathcal{G}) - R f_R] + \nabla_i \nabla_j f_R - g_{ij}\Box f_R + f_{\mathcal{G}}(-2RR_{ij} + 4R_{ik}R^k_j \\ &\quad -2R_i^{klm}R_{jklm} + 4g^{kl}g^{mn}R_{ikjm}R_{ln}) + 2(\nabla_i\nabla_j f_{\mathcal{G}})R - 2g_{ij}(\Box f_{\mathcal{G}})R + 4(\Box f_{\mathcal{G}})R_{ij} \\ &\quad -4(\nabla_k\nabla_i f_{\mathcal{G}})R^k_j - 4(\nabla_k\nabla_j f_{\mathcal{G}})R^k_i + 4g_{ij}(\nabla_k\nabla_l f_{\mathcal{G}})R^{kl} - 4(\nabla_l\nabla_n f_{\mathcal{G}})g^{kl}g^{mn}R_{ikjm} \end{aligned} \tag{2}$$

[1] M. De Laurentis et al., Phys. Rev. D **91**, 083531 (2015)

[2] S.D. Odintsov et al., Nucl. Phys. B **938**, 935 (2019)

Field Equations:

- We consider the spatially flat FLRW metric with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{3}$$

then R and $\mathcal{G}$ become

$$R = 6(\dot{H} + 2H^2) \qquad \mathcal{G} = 24H^2(\dot{H} + H^2) \tag{4}$$

We obtain the field equation from Eq. (2) and Eq. (3)

$$3H^2 f_R = \kappa^2 \rho + \frac{1}{2}[R f_R + \mathcal{G} f_{\mathcal{G}} - f(R,\mathcal{G})] - 3H\dot{f}_R - 12H^3\dot{f}_{\mathcal{G}} \tag{5}$$

$$2\dot{H} f_R + 3H^2 f_R = -\kappa^2 p + \frac{1}{2}[R f_R + \mathcal{G} f_{\mathcal{G}} - f(R,\mathcal{G})] - 2H\dot{f}_R - \ddot{f}_R - 4H^2\ddot{f}_{\mathcal{G}} - 8H\dot{H}\dot{f}_{\mathcal{G}} - 8H^3\dot{f}_{\mathcal{G}} \tag{6}$$

- Here we use $f(R,\mathcal{G}) = R + \mathcal{F}(R,\mathcal{G})$

$$3H^2 = \kappa^2(\rho + \rho_{\text{eff}}), \qquad 3H^2 + 2\dot{H} = -\kappa^2(p + p_{\text{eff}}) \tag{7}$$

Observational Constraints:

- The χ^2_{Hubble} value for the observational Hubble parameter data can be expressed as[3]

$$\chi^2_{\text{Hubble}} = \sum_{i=1}^{32} \frac{[H_{th}(z_i) - H_{\text{obs}}(z_i)]^2}{\sigma_i'^2} \tag{8}$$

- In a sample of 1701 SNe Ia from the Pantheon$^+$ study, the χ^2_{SNe} function is provided by[4]

$$\chi^2_{\text{SNe}} = \sum_{i=1}^{1701} \frac{[\mu_{th}(z_i) - \mu_{\text{obs}}(z_i)]^2}{\sigma_i'^2} \tag{9}$$

- $H^2(z) = H_0^2 E(z)$

  We use the following functional form for $E(z)$ [Lemos et al.[5]]

$$E(z) = [A(1+z)^3 + B + Cz + Dln(1+z)] \tag{10}$$

[3] M. Moresco et al., *Liv. Rev. Rel.* **25**, 6 (2022).
[4] D. Brout et al., *APJ* **938**, 110 (2022).
[5] P. Lemos et al., *MNRAS* **483**, 4803 (2018).

- The contour plots with $1-\sigma$ and $2-\sigma$ errors for the parameters H_0, A, B, C and D.

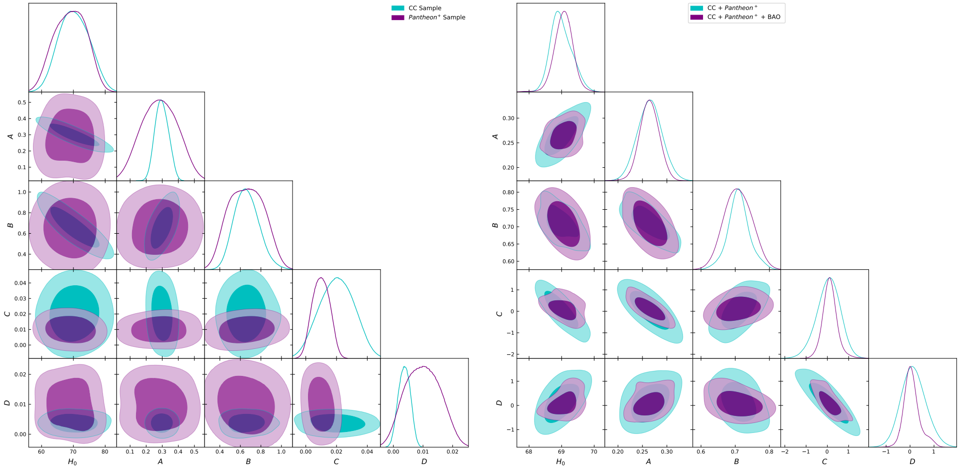

Figure: I

Coefficients	CC Sample	*Pantheon*$^+$	CC + *Pantheon*$^+$	CC + *Pantheon*$^+$ + BAO
H_0	70.2 ± 4.6	69.1 ± 4.8	$68.69^{+0.67}_{-0.59}$	$69.26^{+0.57}_{-0.53}$
A	0.297 ± 0.04	0.28 ± 0.11	$0.285^{+0.050}_{-0.048}$	$0.264^{+0.039}_{-0.036}$
B	$0.66^{+0.11}_{-0.13}$	0.64 ± 0.16	$0.689^{+0.071}_{-0.067}$	$0.698^{+0.070}_{-0.071}$
C	0.0099 ± 0.0053	0.02 ± 0.011	$0.012^{+0.98}_{-1.1}$	0.012 ± 0.71
D	0.0037 ± 0.0019	0.0099 ± 0.056	$0.014^{+1.1}_{-0.99}$	$0.0025^{+0.81}_{-0.61}$

Table: These marginalized constraints are based on the CC samples, *Pantheon*$^+$ samples, and BAO datasets.

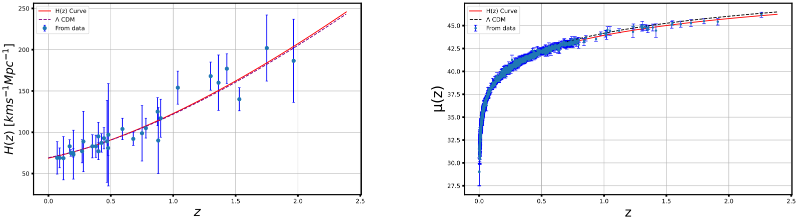

Figure: II

- Here we consider $\mathcal{F}(R,\mathcal{G}) = \alpha R^2 \mathcal{G}^\beta$

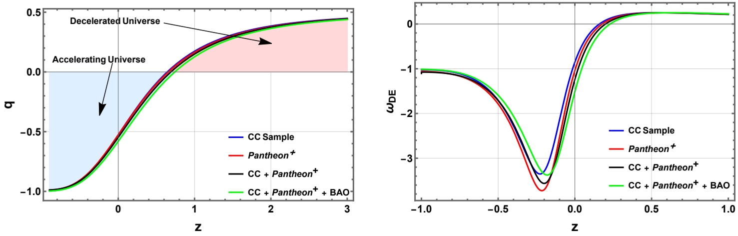

Figure: III

Parameters	CC Sample	*Pantheon*$^+$	CC + *Pantheon*$^+$	CC + *Pantheon*$^+$ + BAO
q	-0.526 ($z_t \approx 0.636$)	-0.529 ($z_t \approx 0.656$)	-0.548 ($z_t \approx 0.691$)	-0.579 ($z_t \approx 0.74$)
ω	-0.8478	-1.02	-1.224	-1.47

Table: Present value of deceleration and EoS parameter based on the CC samples, *Pantheon*$^+$ samples, and BAO datasets.

Stability Analysis of the Model

- Let us define new variable:

$$u_1 = \frac{\dot{f}_R}{H f_R}, \quad u_2 = \frac{f}{6H^2 f_R}, \quad u_3 = \frac{R}{6H^2}, \quad u_4 = \frac{\mathcal{G} f_{\mathcal{G}}}{6H^2 f_R}, \quad u_5 = \frac{4H \dot{f}_{\mathcal{G}}}{f_R}, \quad u_6 = \frac{\kappa^2 \rho_r}{3H^2 f_R}, \quad u_7 = \frac{\kappa^2 \rho_m}{3H^2 f_R}$$

- The dynamical system is

$$\begin{aligned}
\frac{du_1}{dN} &= \frac{\ddot{f}_R}{f_R H^2} - u_1^2 - u_1 \frac{\dot{H}}{H^2}, \\
\frac{du_2}{dN} &= \frac{\dot{f}}{6F_R H^3} - u_1 u_2 - 2u_2 \frac{\dot{H}}{H^2}, \\
\frac{du_3}{dN} &= \frac{\dot{R}}{6H^3} - 2u_3 \frac{\dot{H}}{H^2}, \\
\frac{du_4}{dN} &= \frac{\dot{\mathcal{G}}}{\mathcal{G} H} u_4 + \frac{\mathcal{G}}{24H^4} u_5 - u_1 u_4 - 2u_4(u_3 - 2), \\
\frac{du_5}{dN} &= u_5 \frac{\dot{H}}{H^2} + 4\frac{\ddot{f}_{\mathcal{G}}}{F_R} - u_1 u_5 \\
\frac{du_6}{dN} &= -2u_3 u_6 - u_1 u_6, \\
\frac{du_7}{dN} &= -u_7 \left(3 + u_1 + 2\frac{\dot{H}}{H^2}\right)
\end{aligned}$$

Stability Analysis of the Model

$$u_3 = 2u_2, \qquad u_5 = \frac{u_4}{u_3 - 1}\left[2u_1 + \frac{\beta - 1}{u_3 - 1}\left[2(u_3 - 3)^2 + u_1 u_3\right]\right] \tag{11}$$

Using these relations and the constraint, the system can be reduced to a set of four equations as

$$\frac{du_3}{dN} = u_1 u_3 - 2u_3(u_3 - 2), \tag{12}$$

$$\frac{du_4}{dN} = \frac{\beta u_4}{u_3 - 1}\left[2(u_3 - 3)^2 + u_1 u_3\right] + u_1 u_4 - 2u_4(u_3 - 2), \tag{13}$$

$$\frac{du_6}{dN} = -2u_3 u_6 - u_1 u_6, \tag{14}$$

$$\frac{du_7}{dN} = -u_7(2u_3 + u_1 - 1) \tag{15}$$

where

$$u_1 = \frac{-1 + \frac{3}{2}u_3 + u_6 + u_7 + u_4 - 2(\beta - 1)\frac{(u_3-2)^2}{(u_3-1)^2}u_4}{1 + \frac{u_4}{(u_3-1)}\left[2 + u_3\frac{(\beta-1)}{(u_3-1)}\right]} \tag{16}$$

C.P.	u_3	u_4	u_6	u_7	Exists for
$\mathcal{P}_1$	0	0	1	0	always
$\mathcal{P}_2$	0	$\frac{1-u_6}{5}$	u_6	0	$3+2u_6 \neq 0,\ \beta = \frac{1}{2}$
$\mathcal{P}_3$	2	-2	0	0	$-1+4\beta \neq 0$
$\mathcal{P}_4$	0	u_4	0	0	$-1+u_4 \neq 0, -1+2u_4 \neq 0, \beta = \frac{-3-u_4}{8(-1+u_4)}$
$\mathcal{P}_5$	u_3	$\frac{1}{2}(-6+u_3)$	0	0	$-1+u_3 \neq 0, -2+u_3 \neq 0, 14-12u_3+3u_3^2 \neq 0, \beta = 0$

Table: The critical points of the dynamical system. The coordinates of the critical points: (u_3, u_4, u_6, u_7).

C.P.	Ω_m	Ω_r	Ω_{DE}	q	ω_{eff}
$\mathcal{P}_1$	0	1	0	1	$\frac{1}{3}$
$\mathcal{P}_2$	0	u_6	$1-u_6$	1	$\frac{1}{3}$
$\mathcal{P}_3$	0	0	1	-1	-1
$\mathcal{P}_4$	0	0	1	1	$\frac{1}{3}$
$\mathcal{P}_5$	0	0	1	$1-u_3$	$\frac{1}{3}(1-2u_3)$

Table: Density, deceleration and EoS parameter.

C.P.	Acceleration equation	Phase of the Universe	Stability condition
$\mathcal{P}_1$	$\dot{H}=-2H^2$	$a(t)=t_0(2t+c_1)^{\frac{1}{2}}$	Unstable
$\mathcal{P}_2$	$\dot{H}=-2H^2$	$a(t)=t_0(2t+c_1)^{\frac{1}{2}}$	Unstable
$\mathcal{P}_3$	$\dot{H}=0$	$a(t)=t_0e^{c_1t}$	Stable
$\mathcal{P}_4$	$\dot{H}=-2H^2$	$a(t)=t_0(2t+c_1)^{\frac{1}{2}}$	Unstable
$\mathcal{P}_5$	$\dot{H}=(-2+x_7)H^2$	$a(t)=t_0\left((2-x_7)t+c_1\right)^{\frac{1}{2-x_7}}$	Stable

Table: Phase of the Universe.

Eigenvalues

C.P.	Eigenvalues
$\mathcal{P}_1$	$\{4, -1, 1, -4(-1+2\beta)\}$
$\mathcal{P}_2$	$\{0, \frac{-5(-1+2u_6)}{3+2u_6}, 1, 4\}$
$\mathcal{P}_3$	$\{-4, -3, \frac{3-12\beta-\sqrt{9-136\beta+400\beta^2}}{2(-1+4\beta)}, \frac{3-12\beta+\sqrt{9-136\beta+400\beta^2}}{2(-1+4\beta)}\}$
$\mathcal{P}_4$	$\{\frac{4u_4}{(-1+u_4)(-1+2u_4)}, \frac{-3+u_4}{-1+u_4}, \frac{2(-1+3u_4)}{-1+u_4}, \frac{1-7u_4+10u_4^2}{(-1+u_4)(-1+2u_4)}\}$
$\mathcal{P}_5$	$\{0, \frac{-6(1-2u_3+u_3^2)}{14-12u_3+3u_3^2}, -4(-1+u_3), (5-4u_3)\}$

Table: Equivalent eigenvalues for fixed points.

2-D Phase Potrait

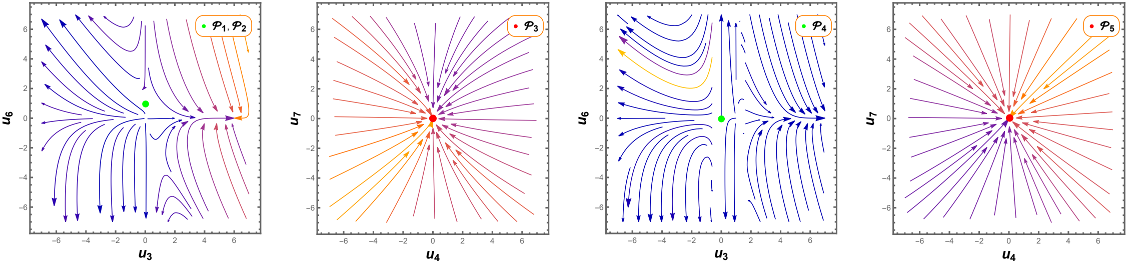

Figure: 2D Phase Portrait for Critical Points $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$.

Evolution of Density Parameters

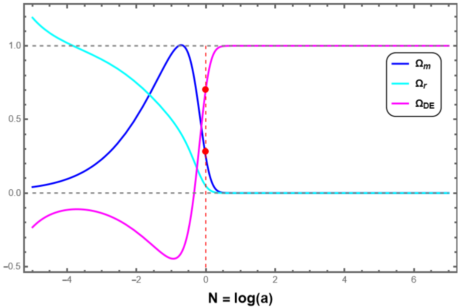

Figure: Evolution of density parameters for the initial conditions: $u_3 = 10^{-9.45}$, $u_4 = 0.01$, $u_6 = 1.28999$, $u_7 = 0.448 \times 10^{-1.2}$. The current densities obtained are: $\Omega_m \approx 0.28$, $\Omega_{DE} \approx 0.679$, and $\Omega_r \approx 0.047$

Results and Discussion

- We assume a functional form of $\mathcal{F}(R,\mathcal{G})$ and Hubble parameter, $\mathcal{F}(R,\mathcal{G}) = \alpha R^2 \mathcal{G}^\beta$ and $H^2(z) = H_0^2[A(1+z)^3 + B + Cz + Dln(1+z)]$. The Hubble parameter formula coefficients have been constrained using the CC sample, the largest *Pantheon*$^+$ and the BAO dataset.
- We derive the deceleration parameter q and EoS parameter from our constrained values. For CC, *Pantheon*$^+$, CC + *Pantheon*$^+$ and CC + *Pantheon*$^+$ + BAO data, the transition redshifts are $z_t = 0.636$, $z_t = 0.656$, $z_t = 0.691$ and $z_t = 0.74$, respectively.Our result for the effective EoS parameter at $z = 0$ is -0.8478, -1.02, -1.224 and -1.47 for CC, *Pantheon*$^+$, CC + *Pantheon*$^+$ and CC + *Pantheon*$^+$ + BAO datasets respectively, is mostly in line with the current observational findings.
- For the dynamical analysis of our proposed model $\mathcal{F}(R,\mathcal{G}) = \alpha R^2 \mathcal{G}^\beta$, there are a total of five critical points obtained, two of which $(\mathcal{P}_3, \mathcal{P}_5)$ are stable and five $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_4)$ of which are unstable. The trajectory behavior indicates that the unstable critical points act as release points while the stable ones act as attractor points.
- Evolution of density parameters was plotted. The current densities obtained are: $\Omega_m \approx 0.28$, $\Omega_{DE} \approx 0.679$, and $\Omega_r \approx 0.047$. Radiation dominance is shown in Figure at the beginning, followed by a brief phase of matter dominance and, at the end, the de-Sitter phase.

Classical and Quantum Gravity DOI: 10.1088/1361-6382/acfc0f

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Thank You!