Resonant generation of high-order harmonics in nonlinear electrodynamics in cavities: Quantum perturbative approach.

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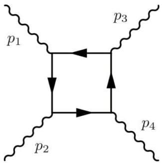
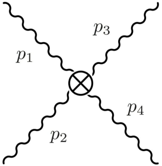
Nonlinear electrodynamics: Euler-Heisenberg Lagrangian

Interactions with virtual electrons are integrated out (the limit $p \ll m_e$),





Effective Euler-Heisenberg Lagrangian

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{90 m_e^4} \left((F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} \left(F_{\mu\nu} \widetilde{F}^{\mu\nu} \right)^2 \right) + O(\alpha^4).$$

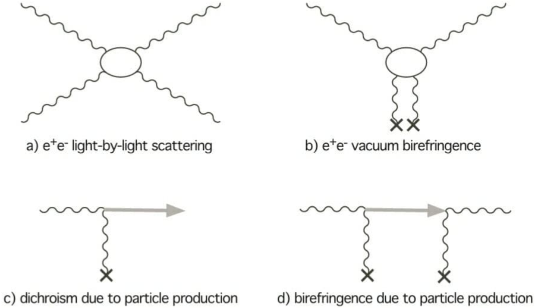
H.Euler and B. Kockel (1935), W.Heisenberg and H.Euler (1936)

No experimental proof in 2023!

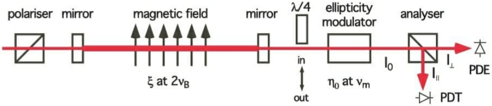
- The contribution of the same type to $\mathcal{L}_{eff}$ from new physics particles: scalars and pseudoscalars (axions)

Effects of nonlinear ED

Credit: PVLAS coll. EPJ C (2016), arXiv:1510.08052



Birefringence in external magnetic field: PVLAS experiment

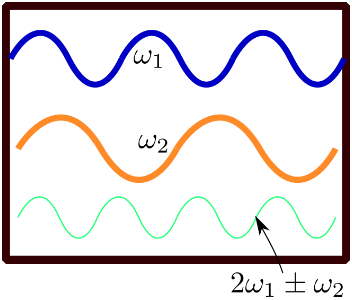


PVLAS result: No new physics particles in a given region of parameters – 1 order of magnitude below Euler-Heisenberg sensitivity

Photon-photon scattering in radiofrequency cavities

Nonlinear theory with 4-photon interaction
EM modes in radio-frequency cavities: possibility for generation of a mode of combined frequency G. Brodin, M. Marklund, L. Stenflo. PRL (2001)

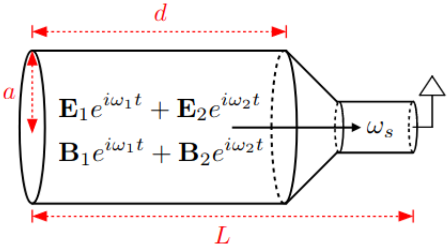
- pump modes: ω_1, ω_2
  $E_p = E_1\, e^{i\omega_1 t} + E_2\, e^{i\omega_2 t}$
- signal modes:
  $2\omega_{1(2)} \pm \omega_{2(1)}$, $3\omega_{1(2)}$.
- Resonant amplification of signal modes
- Q up to $10^{10} - 10^{12}$ in superconducting cavities



Partial solutions for $\omega_s = 2\omega_1 - \omega_2$
D. Eriksson, G. Brodin, M. Marklund, L. Stenflo. PRA (2004)
hard to make an experiment at the technology level of 2004

Experimental projects

Z. Bogorad, A. Hook, Y. Kahn, Y. Soreq. PRL (2020)



(focused on axion searches)

$\omega_1 = TE_{011}$, $\omega_2 = TM_{010}$
$\omega_s = 2\omega_1 - \omega_2 = TM_{020}$,
if $d = 3.112a$

(SQMS, Fermilab) B. Giaccone et al. arXiv:2207.11346 (2022)

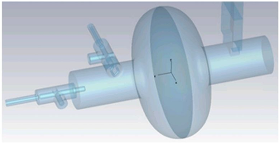
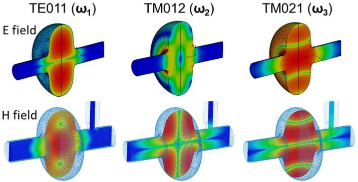


FIG. 9. RF geometry for the three-mode axion search. The design of this cavity is still under study.



Calculation of the signal mode resonant generation

Two approaches

- (A) Solve the classical EoMs obtained from EFT Lagrangian (EFT = nonlinear ED). Perturbative limit in solving differential equations → numerical solution for given sets of modes
  Resonance: a signal mode grows with time. Stops at $t \sim Q \cdot T$.
  D. Eriksson, G. Brodin, M. Marklund, L. Stenflo. PRA (2004)
  Z. Bogorad, A. Hook, Y. Kahn, Y. Soreq. PRL (2020)
- (B) Compute the perturbative amplitude for quantum scattering process

Open points in works on (A):

- Only $2\omega_1 - \omega_2$ signal mode considered previously, what happened with $3\omega_1$ or $2\omega_1 + \omega_2$?
- K. Shibata. EPJ D (2020) No 3rd harmonics ($3\omega_1$) generation in 1D cavity
- I. Kopchinskii, P.S. PRA (2022) Analytical solutions: ($3\omega_1$) and ($2\omega_1 + \omega_2$) are not generated in 1D and 3D rectangular cavities for any sets of modes (combinatoric proof)

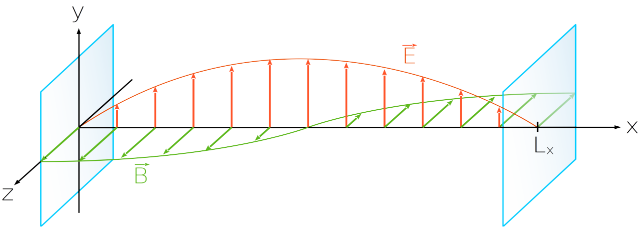
Open questions in "classical" approach (A)

Open questions:

- What is the reason for the absence of the resonance for $(3\omega_1)$ and $(2\omega_1 + \omega_2)$? The 3rd harmonics is resonantly amplified in $\lambda\phi^4$ theory!
- Res. generation of $2\omega_1 - \omega_2$: what is the quantum description? Naively, final state of $3 \to 1$ particle process should have energy $2\omega_1 + \omega_2$.
- Is it OK to use "classical" effective theory in case of $N \sim 1$ signal quanta in the final state?

Hope that the quantum amplitude calculation will shed a light on these points

Quantization of free EM field in 1D cavity



Boundary conditions: $E \cdot \tau \big|_{x=0,L_x} = 0, \quad B \cdot n \big|_{x=0,L_x} = 0 \quad \rightarrow \quad A \cdot \tau \big|_{x=0,L_x} = 0, \quad A_0 = 0$

Mode decomposition: $i = y, z$ – 2 polarizations

$$A_i(t,x) = A_i^+(t,x) + A_i^-(t,x), \qquad A_i^{\pm}(t,x) = \sum_{n=1}^{\infty} \frac{\sin(k_n x)}{\sqrt{k_n L_x}} \, a_i^{\pm}(k_n)\, e^{\pm i k_n t}, \qquad k_n = \frac{\pi n}{L_x}$$

Quantization: $[a_i^-(k_n), a_j^+(k_{n'})] = \delta_{ij}\, \delta_{nn'} \qquad a^+(k_n)|0\rangle = |1_n\rangle \qquad \frac{\left(a^+(k_n)\right)^m |0\rangle}{\sqrt{m!}} = |m_n\rangle$

Connection with plane waves: $|1_n\rangle = \frac{1}{2i}\left(|k_n\rangle - |-k_n\rangle\right)$

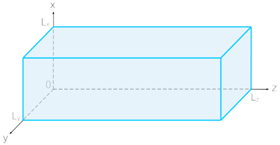
Quantization in 3D rectangular cavity

Details of canonical quantization: Chenaran, Shirzad arXiv:1311.0361

Decompose $A(t,\mathbf{x})$ into TE- and TM-modes $\mathcal{A}^{\lambda}_{npq}(\mathbf{x})$, $\quad \lambda \in \{\mathrm{TE}, \mathrm{TM}\}$

$$A(t,\mathbf{x}) = A^{+}(t,\mathbf{x}) + A^{-}(t,\mathbf{x}), \qquad A^{\pm}(t,\mathbf{x}) = \sum_{\lambda,npq} a^{\lambda\pm}_{npq}\,\mathcal{A}^{\lambda}_{npq}(\mathbf{x})\,\frac{e^{\pm i\omega_{npq}t}}{\sqrt{2\omega_{npq}}}$$

$$\boldsymbol{\mathcal{A}}^{\mathrm{TM}}_{npq}(\mathbf{r}) = \frac{N^{TM}}{\omega_{npq}}\begin{pmatrix} \frac{k_x k_z}{\sqrt{k_x^2+k_y^2}}\cos(k_x x)\sin(k_y y)\sin(k_z z) \\ \frac{k_y k_z}{\sqrt{k_x^2+k_y^2}}\sin(k_x x)\cos(k_y y)\sin(k_z z) \\ -\sqrt{k_x^2+k_y^2}\sin(k_x x)\sin(k_y y)\cos(k_z z)\end{pmatrix}, \quad \boldsymbol{\mathcal{A}}^{\mathrm{TE}}_{npq}(\mathbf{r}) = N^{TE}\begin{pmatrix} +\frac{k_y}{\sqrt{k_x^2+k_y^2}}\cos(k_x x)\sin(k_y y)\sin(k_z z) \\ -\frac{k_x}{\sqrt{k_x^2+k_y^2}}\sin(k_x x)\cos(k_y y)\sin(k_z z) \\ 0\end{pmatrix}$$



$$k_x = \frac{\pi n}{L_x},\, k_y = \frac{\pi p}{L_y},\, k_z = \frac{\pi q}{L_z}, \qquad \omega_{npq} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

Quantization: $[a^{\lambda-}_{npq}, a^{\lambda'+}_{n'p'q'}] = \delta_{\lambda\lambda'}\,\delta_{nn'}\,\delta_{pp'}\,\delta_{qq'} \qquad a^{\lambda+}_{npq}|0\rangle = |1^{\lambda}_{npq}\rangle, \quad \frac{\left(a^{\lambda+}_{npq}\right)^m|0\rangle}{\sqrt{m!}} = |m^{\lambda}_{npq}\rangle$

$|1^{\lambda}_{npq}\rangle = 8$ plane waves

Signal mode generation as QFT perturbative process

- Signal mode generation in nonlinear electrodynamics = nonlinear interaction between quanta of pump modes. Elementary process includes 4 quanta.
- $P \propto |\langle f|U(t_f - t_i)|i\rangle|^2$
- $|i\rangle$ and $|f\rangle$ are not plane waves but cavity eigenmodes (=linear combinations of plane waves, $|i\rangle = \Sigma_n c_n^i |k_n\rangle$). S-matrix formalism is still applicable since the theory is linear. 2 options:
  - $P \propto \Sigma_{nm} (c_m^f)^* c_n^i \langle k_m|\mathrm{S}|k_n\rangle$ – OK if small number of terms in a sum
  - $P \propto |\langle f|S|i\rangle|^2$ in general case
- $T_{fi} = 2\pi\delta\left(\Sigma_i \omega_i - \omega_f\right) M_{fi} = 2\pi\delta(0) M_{fi}$ for the resonance condition
  - No dissipation: $2\pi\delta(0) \to T_{int}$ (see any QFT textbook)
  - Dissipation: $2\pi\delta(0) \to T_{1/2diss} = \frac{Q}{\omega_f}$
- Mean number of signal photons in steady regime $N_s = \frac{Q^2}{\omega_f^2} |M_{fi}|^2$.

Wick contractions for cavity modes

1D cavity

$$\overbrace{a^{-}_{i,n}\, E}_j(t,x) = \delta_{ij}\, i \sqrt{\frac{\omega_n}{V}} \sin(k_n x)\, e^{i\omega_n t},$$

$$\overbrace{a^{-}_{z,n}\, B}_y(t,x) = -\sqrt{\frac{\omega_n}{V}} \cos(k_n x)\, e^{i\omega_n t} = -\overbrace{a^{-}_{y,n}\, B}_z(t,x),$$

3D cavity

$$\overbrace{a^{\lambda -}_{npq}\, \mathbf{E}}(t,\mathbf{r}) = i \sqrt{\frac{\omega_{npq}}{2V}} \boldsymbol{\mathcal{A}}^{\lambda}_{npq}(\mathbf{r})\, e^{i\omega_{npq} t}, \qquad \overbrace{a^{\lambda -}_{npq}\, \mathbf{B}}(t,\mathbf{r}) = \frac{1}{\sqrt{2\omega_{npq} V}} \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}^{\lambda}_{npq}(\mathbf{r})\, e^{i\omega_{npq} t}.$$

$3 \to 1$ merging process in 1D

2 quanta of cavity mode ω_n + 1 quantum ω_p, arbitrary polarizations

$$|i\rangle = \left|1^i_n\right\rangle \otimes \left|1^j_n\right\rangle \otimes \left|1^l_p\right\rangle = a^+_{i,n} a^+_{j,n} a^+_{l,p} \left|0\right\rangle , \qquad |f\rangle = \left|1^s_{2n+p}\right\rangle = a^+_{s,2n+p} \left|0\right\rangle$$

$$T_{fi} = \langle f|S|i\rangle = i\frac{\alpha^2}{90 m_e^4} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \iint \mathrm{d}S \int\limits_0^{L_x} \mathrm{d}x \, \langle f|(\mathbf{EE})^2 - 2\mathbf{B}^2\mathbf{E}^2 + (\mathbf{BB})^2 + 7(\mathbf{BE})^2|i\rangle \, ,$$

$$\langle(\mathbf{EE})^2\rangle = \langle(\mathbf{BB})^2\rangle = -\frac{1}{2}\langle\mathbf{B}^2\mathbf{E}^2\rangle = \\ = 2\pi\delta(0)\frac{\sqrt{(2n+p)n^2 p}\pi^2}{L_x^3}\left[\delta_{ij}\delta_{ls}(1+2\delta_{is}) + (1-\delta_{ls})(1-\delta_{ij})\right] ,$$

and $\langle(\mathbf{EB})^2\rangle = 0$.
Finally, vanishes. What is the reason?

$3 \to 1$ merging. Plane wave decomposition approach

The 4-point amplitude $\langle f|\mathrm{S}|i\rangle = \left\langle 1^s_{2n+p}\middle|\mathrm{S}\middle|1^i_n, 1^j_n, 1^l_p\right\rangle$ decomposes to 16 plane wave amplitudes

$$\frac{(-1)^{\pm}}{(2i)^4}\left\langle \pm k^s_{2n+p}\middle|\mathrm{S}\middle|\pm k^i_n, \pm k^j_n, \pm k^l_p\right\rangle,$$

where $(-1)^{\pm} = 1$ in case of even number of sign $+$ in the amplitude and -1 otherwise. The energy-momentum conservation shows that 14 amplitudes are zero, the remained ones are

$$\frac{1}{(2i)^4}\left\langle k^s_{2n+p}\middle|\mathrm{S}\middle|k^i_n, k^j_n, k^l_p\right\rangle = \frac{1}{(2i)^4}\left\langle -k^s_{2n+p}\middle|\mathrm{S}\middle|-k^i_n, -k^j_n, -k^l_p\right\rangle$$

- $\left\langle k^s_{2n+p}\middle|\mathrm{S}\middle|k^i_n, k^j_n, k^l_p\right\rangle$ — amplitude for $3 \to 1$ parallel plane wave merging. Lorentz scalar, depend only on scalar products $(k^j_\mu k^l_\mu)$, vanishes for parallel momenta
- The same idea for $(3\omega_1)$ and $(2\omega_1 + \omega_2)$ in 3D

$2 \to 2$ scattering

- $2\omega_1 - \omega_2$ cannot be final state of $3 \to 1$ merging due to the energy conservation.
- Classical pump modes are not states with fixed number of particles but the coherent states $\left|\xi^{\lambda}_{npq}\right\rangle = e^{-\frac{|\xi|^2}{2}} \sum\limits_{i=0}^{\infty} \frac{\xi^i}{i!} \left(a^{+\lambda}_{npq}\right)^i |0\rangle$
- Idea: Elementary process – $2 \to 2$ scattering, sum into coherent states
- 2TE011 $\to$ TM110+TM130 for concreteness

Initial and final states for $2 \to 2$,

$$|i\rangle = \left|2^{\mathrm{TE}}_{011}\right\rangle = \frac{1}{\sqrt{2}}\left(a^{\mathrm{TE}+}_{011}\right)^2 |0\rangle\,, \qquad |f\rangle = \left|1^{\mathrm{TM}}_{110}\right\rangle \otimes \left|1^{\mathrm{TM}}_{130}\right\rangle = a^{\mathrm{TM}+}_{110} a^{\mathrm{TM}+}_{130} |0\rangle\,.$$

The matrix element for $2 \to 2$,

$$\mathrm{M}_{2\to 2} = i\frac{2\alpha^2}{45 m_e^4}\int\limits_V \mathrm{d}^3x \,\langle f|\,\mathbf{E}^4 - 2\mathbf{B}^2\mathbf{E}^2 + \mathbf{B}^4 + 7(\mathbf{BE})^2\,|i\rangle =$$

$$= \left\langle \mathbf{E}^4 \right\rangle - 2\left\langle \mathbf{B}^2\mathbf{E}^2 \right\rangle + \left\langle \mathbf{B}^4 \right\rangle + 7\left\langle (\mathbf{BE})^2 \right\rangle,$$

$2 \to 2$ scattering, 2TE011 $\to$ TM110+TM130

Cavity dimensions $L_x : L_y : L_z = 1 : 1 : r$
Energy conservation for $2 \to 2$: $\quad 2\omega_{011} = \omega_{110} + \omega_{130}$
Nonzero result for $r = \sqrt{\sqrt{5} - 2}$

$$\mathrm{M}_{2\to 2} = \left\langle \mathbf{E}^4 \right\rangle - 2\left\langle \mathbf{B}^2\mathbf{E}^2 \right\rangle + \left\langle \mathbf{B}^4 \right\rangle + 7\left\langle (\mathbf{BE})^2 \right\rangle,$$

$$\left\langle \mathbf{E}^4 \right\rangle = -\frac{1}{8V}\sqrt{\omega_{011}^2\omega_{110}\omega_{130}}, \qquad \left\langle \mathbf{B}^4 \right\rangle = \frac{1}{4V}\frac{\pi^4}{L_z^4}r^2\frac{2r^2-3}{\sqrt{\omega_{011}^2\omega_{110}\omega_{130}}},$$

$$\left\langle \mathbf{B}^2\mathbf{E}^2 \right\rangle = \frac{1}{8V}\frac{\pi^2}{L_z^2}\left((r^2-1)\sqrt{\frac{\omega_{110}\omega_{130}}{\omega_{011}^2}} - 4r^2\sqrt{\frac{\omega_{011}^2}{\omega_{110}\omega_{130}}}\right),$$

$$\left\langle (\mathbf{BE})^2 \right\rangle = \frac{1}{8V}\frac{\pi^2}{L_z^2}r^2\left(\sqrt{\frac{\omega_{110}\omega_{130}}{\omega_{011}^2}} - 3\sqrt{\frac{\omega_{011}^2}{\omega_{110}\omega_{130}}}\right).$$

Even single pump mode (TE011) produce two signal modes (TM110 and TM130) by a nonlinear interaction
Bose enhancement if TM110 already excited

$2\omega_1 - \omega_2$ generation by coherent states

the initial and final states read,

$$|i\rangle = \left|\xi_{011}^{\text{TE}}\right\rangle \otimes \left|\eta_{110}^{\text{TM}}\right\rangle = e^{-\frac{|\xi|^2+|\eta|^2}{2}} \sum_{i,j=0}^{\infty} \frac{\xi^i \eta^j}{i!j!} \left(a_{011}^{\mathrm{TE}+}\right)^i \left(a_{110}^{\mathrm{TM}+}\right)^j |0\rangle \,,$$

$$|f\rangle = \left|\xi_{011}^{\text{TE}}\right\rangle \otimes \left|\eta_{110}^{\text{TM}}\right\rangle \otimes \left|1_{130}^{\text{TM}}\right\rangle = |i\rangle \otimes \left|1_{130}^{\text{TM}}\right\rangle = a_{130}^{\text{TM}+} |i\rangle \,.$$

The parameters ξ, η are associated with the mean number of quanta in the pump modes,

$$\langle N_{\text{TE}_{011}}\rangle = |\xi|^2, \qquad \langle N_{\text{TM}_{110}}\rangle = |\eta|^2.$$

$$\text{M}_{\text{coh}} = \xi^2 \eta^* \times \sqrt{2}\, \text{M}_{2\to 2}.$$

$$P_{\text{coh}} = 2 \langle N_{\text{TE}_{011}}\rangle^2 \cdot \langle N_{\text{TM}_{110}}\rangle \cdot P_{2\to 2}.$$

$$\langle N_s\rangle = P_{\text{coh}} = G_1^2 \times \left(\frac{\alpha^2}{90 m_e^2}\right)^2 Q^2 F_0^6 L_z^4,$$

$$G_1^2 = \frac{4}{(10)^{3/2}\pi r^3 (1+r^2)^2} \left[5 + 2\sqrt{5} - \frac{7}{4}\left(\sqrt{1+r^2} + \sqrt{2}r\right)^2\right]^2.$$

The same result as in “classical” approach I.Kopchinskii, P.S. PRA (2022)

Conclusions

- We developed technique for perturbative calculations in cavities in nonlinear electrodynamics
- $3\omega_1$ and $2\omega_1 + \omega_2$ do not resonate due to plane wave decomposition, LI and photon zero mass
- $2 \to 2$ is a crucial elementary process for $2\omega_1 - \omega_2$ resonant generation, number of quanta does not conserve
- Straightforward generalization to: other geometries, other initial states (squeezed etc), mean values of other operators etc..

Thank you for your attention![1]

[1]The talk is supported by RSF grant 21-72-1015