Photon Polarization Operator in External Electromagnetic Field with Account of Virtual-Fermion AMM

Alexander Parkhomenko

P. G. Demidov Yaroslavl State University, Yaroslavl, Russia

International Conference on Particle Physics and Cosmology
Yerevan, Armenia, 2 — 7 October, 2023

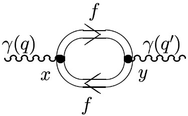
in collaboration with Alexandra Dobrynina, Ilya Karabanov, & Lubov Vassilevskaya

Photon Polarization Operator

- Photon polarization operator is typical example of two-point correlation function
- Lagrangian of spinor QED

$$\mathcal{L}_{\rm QED}(x) = eQ_f \left[\bar{f}(x)\gamma_\mu f(x)\right] A^\mu(x)$$

- Matrix element of $\gamma \to \gamma$ transition



$$\mathcal{M}_{\gamma\to\gamma} = -i\,\varepsilon'^{*}_{\mu}(q)\,\mathcal{P}^{\mu\nu}(q)\,\varepsilon_\nu$$

- $\mathcal{P}^{\mu\nu}(q)$ is two-point correlator of two vector currents
- Photon dispersion relations follow from the equations

$$q^2 - \Pi^{(\lambda)}(q) = 0 \quad (\lambda = 1, 2, 3)$$

- $\Pi^{(\lambda)}(q)$ are eigenvalues of the photon polarization operator
- In an external background field, corresponding modification of fermion propagator should be taken into account

Basic Tensors in Presence of Magnetic Field

- Minkowski space filled with external magnetic field is divided into two subspaces:
  - Euclidean with the metric tensor $\Lambda_{\mu\nu} = (\varphi\varphi)_{\mu\nu}$; plane orthogonal to the field strength vector
  - Pseudo-Euclidean with the metric tensor $\tilde{\Lambda}_{\mu\nu} = (\tilde{\varphi}\tilde{\varphi})_{\mu\nu}$
  - Metric tensor of Minkowski space $g_{\mu\nu} = \tilde{\Lambda}_{\mu\nu} - \Lambda_{\mu\nu}$
- Dimensionless tensor of the external magnetic field and its dual

$$\varphi_{\alpha\beta} = \frac{F_{\alpha\beta}}{B}\,, \qquad \tilde{\varphi}_{\alpha\beta} = \frac{1}{2}\,\varepsilon_{\alpha\beta\rho\sigma}\varphi^{\rho\sigma}$$

- Arbitrary four-vector $a^\mu = (a_0, a_1, a_2, a_3)$ can be decomposed into two orthogonal components

$$a_\mu = \tilde{\Lambda}_{\mu\nu}a^\nu - \Lambda_{\mu\nu}a^\nu = a_{\parallel\mu} - a_{\perp\mu}$$

- For the scalar product of two four-vectors one has

$$(ab) = (ab)_\parallel - (ab)_\perp$$

$$(ab)_\parallel = (a\tilde{\Lambda}b) = a^\mu\tilde{\Lambda}_{\mu\nu}b^\nu, \quad (ab)_\perp = (a\Lambda b) = a^\mu\Lambda_{\mu\nu}b^\nu$$

Orthogonal Basis Motivated by Magnetic Field

- Correlators having rank non-equal to zero, could be decomposed in some orthogonal set of vectors
- In magnetic field, such a basis naturally exists

$$b_\mu^{(1)} = (q\varphi)_\mu, \qquad b_\mu^{(2)} = (q\tilde{\varphi})_\mu$$
$$b_\mu^{(3)} = q^2 \, (\Lambda q)_\mu - (q\Lambda q)\, q_\mu, \quad b_\mu^{(4)} = q_\mu$$

- Arbitrary vector a_μ can be presented as

$$a_\mu = \sum_{i=1}^{4} a_i \, \frac{b_\mu^{(i)}}{(b^{(i)} b^{(i)})}, \qquad a_i = a^\mu b_\mu^{(i)}$$

- Third-rank tensor $T_{\mu\nu\rho}$ can be decomposed similarly

$$T_{\mu\nu\rho} = \sum_{i,j,k=1}^{4} T_{ijk} \, \frac{b_\mu^{(i)} b_\nu^{(j)} b_\rho^{(k)}}{(b^{(i)} b^{(i)})\,(b^{(j)} b^{(j)})\,(b^{(k)} b^{(k)})},$$

$$T_{ijk} = T^{\mu\nu\rho} b_\mu^{(i)} b_\nu^{(j)} b_\rho^{(k)}.$$

Photon Polarization Operator in Magnetic Field

- $\Pi^{(\lambda)}(q)$ are eigenvalues of the photon polarization operator

$$\mathcal{P}_{\mu\nu}(q) = \sum_{\lambda=1}^{3} \frac{b_\mu^{(\lambda)} b_\nu^{(\lambda)}}{(b^{(\lambda)})^2} \, \Pi^{(\lambda)}(q)$$

- In vacuum, $\mathcal{P}_{\mu\nu}(q)$ has two physical eigenmodes
- In an external constant homogeneous magnetic field, the number of physical eigenmodes is the same
- Eigenvectors are determined by the field strength tensor

$$\varepsilon_\mu^{(1)} = b_\mu^{(1)} / \sqrt{q_\perp^2}, \quad \varepsilon_\mu^{(2)} = b_\mu^{(2)} / \sqrt{q_\parallel^2}$$

- In the magnetic field, $\Pi^{(\lambda)}(q)$ contains both vacuum and field-induced parts (for electron)

$$\Pi^{(\lambda)}(q) = -i\,\mathcal{P}(q^2) - \frac{\alpha}{\pi}\, Y_{VV}^{(\lambda)}$$

- Details on $Y_{VV}^{(\lambda)}$ can be found in A. Kuznetsov & N. Mikheev, Electroweak Processes in External Electromagnetic Fields (Springer, 2013)

Inclusion of Fermion AMM

- Models beyond the SM can produce effective operators at current energies and Pauli Lagrangian density, in particular

$$\mathcal{L}_{\mathrm{AMM}}(x) = -\frac{\mu_f}{2}\left[\bar{f}(x)\sigma_{\mu\nu}f(x)\right]F^{\mu\nu}(x)$$

- For electron, the coupling can be written as $\mu_e = \mu_B a_e$, where $\mu_B = e/(2m_e)$ is Bohr magneton and a_e is electron AMM
- Total Lagrangian of interaction

$$\mathcal{L}_{\mathrm{int}}(x) = \mathcal{L}_{\mathrm{QED}}(x) + \mathcal{L}_{\mathrm{AMM}}(x)$$

- It gives additional contribution to the polarization operator
- Contribution linear in AMM is related with correlator of vector and tensor currents, $\Pi^{(VT)}_{\mu\nu\rho}$
- Contribution quadratic in AMM is determined by correlator of two tensor currents, $\Pi^{(TT)}_{\mu\nu\rho\sigma}$

Generalized Local Fermionic Current

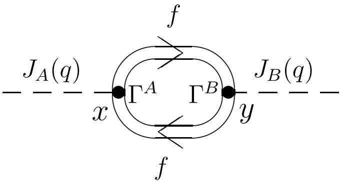
[M. Yu. Borovkov et al., Phys. At. Nucl. 62 (1999) 1601]

- Lagrangian density of local fermion interaction

$$\mathcal{L}_{\rm int}(x) = \left[\bar{f}(x)\Gamma^A f(x)\right] J_A(x)$$

- J_A — generalized current (photon, neutrino current, etc.)
- Γ_A — any of γ-matrices from the set $\{1,\, \gamma_5,\, \gamma_\mu,\, \gamma_\mu\gamma_5, \sigma_{\mu\nu} = i\left[\gamma_\mu, \gamma_\nu\right]/2\}$
- Interaction constants are included into the current J_A

General Case of Two-Point Correlator



- Two-point correlation function of general form

$$\Pi_{AB} = \int d^4X\, \mathrm{e}^{-i(qX)}\, \mathrm{Sp}\left\{S_{\mathrm{F}}(-X)\,\Gamma_A\, S_{\mathrm{F}}(X)\,\Gamma_B\right\}$$

- $S_{\mathrm{F}}(X)$ — gauge and translationally invariant part of the fermion propagator
- $X^\mu = x^\mu - y^\mu$ — integration variable
- Correlations of scalar, pseudoscalar, vector and axial-vector currents were studied by Borovkov et al. [Phys. At. Nucl. 62 (1999) 1601]
- Consider correlations of a tensor current with the other ones

Propagator in the Fock-Schwinger Representation

- General representation of the propagator in magnetic field [J.S. Schwinger, Phys. Rev. 82 (1951) 664]

$$G_{\rm F}(x,y) = {\rm e}^{i\Phi(x,y)}\, S_{\rm F}(x-y)$$

- Translationally and gauge non-invariant phase factor

$$\Phi(x,y) = -eQ_f \int_y^x d\xi^\mu \left[A_\mu(\xi) + \frac{1}{2} F_{\mu\nu}(\xi - y)^\nu\right]$$

- In two-point correlation function phase factors cancel each other

$$\Phi(x,y) + \Phi(y,x) = 0$$

- Gauge and translationally invariant part of a charged fermion propagator ($\beta = eB\,Q_f$)

$$\begin{aligned} S_{\rm F}(X) &= -\frac{i\beta}{2(4\pi)^2} \int_0^\infty \frac{ds}{s^2} \Big\{ (X\widetilde{\Lambda}\gamma)\cot(\beta s) - i(X\widetilde{\varphi}\gamma)\gamma_5 - \\ &- \frac{\beta s}{\sin^2(\beta s)} (X\Lambda\gamma) + m_f s\, [2\cot(\beta s) + (\gamma\varphi\gamma)] \Big\} \times \\ &\times \exp\left(-i\left[m_f^2 s + \frac{1}{4s}(X\widetilde{\Lambda}X) - \frac{\beta\cot(\beta s)}{4}(X\Lambda X)\right]\right) \end{aligned}$$

Correlator of Vector and Tensor Currents

- Vector-tensor (VT) correlator, $\Pi^{(\mathrm{VT})}_{\mu\nu\rho}$, is rank-3 tensor
- Vector-current conservation and antisymmetry of the tensor current leave 18 non-trivial coefficients in the decomposition on basis vectors
- Of them, four coefficients only are independent
- Double-integral representation of coefficients is used

$$\Pi^{(\mathrm{VT})}_{ijk}(q^2, q_\perp^2, \beta) = \frac{1}{4\pi^2} \int_0^\infty \frac{dt}{t} \int_0^1 du\, e^{-i\Omega(t,u)}\, Y^{(\mathrm{VT})}_{ijk}(q^2, q_\perp^2, \beta; t, u)$$

- Phase definition

$$\Omega(t,u) = m_f^2 t - \frac{q_\parallel^2}{4}\, t\,(1-u^2) + q_\perp^2\, \frac{\cos(\beta t u) - \cos(\beta t)}{2\beta \sin(\beta t)}$$

- Integration variables and relation between momenta squared
  $t = s_1 + s_2,\ u = (s_1 - s_2)/(s_1 + s_2);\quad q_\parallel^2 = q^2 + q_\perp^2$

Integrands in Vector-Tensor Correlator

$$Y_{114}^{(\mathrm{VT})}(t,u) = -Y_{141}^{(\mathrm{VT})}(t,u) = -m_f\, q_\perp^2\, q^2\, \frac{\beta t \cos(\beta t u)}{\sin(\beta t)}$$

$$Y_{223}^{(\mathrm{VT})}(t,u) = -Y_{232}^{(\mathrm{VT})}(t,u) = m_f\, q_\perp^2\, (q_\parallel^2)^2\, \frac{\beta t}{\sin(\beta t)} \left[\cos(\beta t) - \cos(\beta t u)\right]$$

$$Y_{224}^{(\mathrm{VT})}(t,u) = -Y_{242}^{(\mathrm{VT})}(t,u) = m_f\, q_\parallel^2\, \frac{\beta t}{\sin(\beta t)} \left[q_\perp^2 \cos(\beta t) - q_\parallel^2 \cos(\beta t u)\right]$$

$$Y_{334}^{(\mathrm{VT})}(t,u) = -Y_{343}^{(\mathrm{VT})}(t,u) = -m_f\, q_\perp^2\, q_\parallel^2\, (q^2)^2\, \frac{\beta t \cos(\beta t u)}{\sin(\beta t)}$$

- Choice of basis vectors is optimal because of vector current conservation $q^\mu \Pi_{\mu\nu\rho}^{(\mathrm{VT})}$
- $Y_{4jk}^{(\mathrm{VT})}$ vanish naturally in this basis
- Antisymmetry in the last two indices is due to antisymmetric tensor current
- Parameters q^2, $q_\perp^2$, and β in $Y_{ijk}^{(\mathrm{VT})}$ are assumed implicitly

VT Contribution to $\gamma \to \gamma$ Amplitude

- Basis vectors are normalized, so $\gamma \to \gamma$ amplitude by itself is required to extract the photon polarization operator
- Vector and tensor currents in momentum space

$$j_V^\mu = -eQ_f \varepsilon'^\mu, \quad j_T^{\nu\rho} = i g_T f^{*\nu\rho} = i g_T \left(q^\nu \varepsilon^{*\rho} - q^\rho \varepsilon^{*\nu} \right)$$

- Relation among the $\gamma \to \gamma$ amplitude and VT correlator

$$\mathcal{M}_{\mathrm{VT}} = -ieQ_f g_T \varepsilon'^\mu \Pi^{(\mathrm{VT})}_{\mu\nu\rho} f^{*\nu\rho}$$

- The $\gamma \to \gamma$ amplitude in explicitly gauge invariant form

$$\mathcal{M}_{\mathrm{VT}} = \frac{eQ_f g_T m_f \beta}{16\pi^2} \int\limits_0^\infty \frac{dt}{\sin(\beta t)} \int\limits_0^1 du\, e^{-i\Omega(t,u)} \times \left\{ \cos(\beta t u) \left(f' f^* \right) + \frac{q_\perp^2}{2q_\parallel^2} \left[\cos(\beta t) - \cos(\beta t u) \right] \left(\tilde{\varphi} f' \right) \left(\tilde{\varphi} f^* \right) \right\}$$

- Used the notation for tensor contractions

$$\left(f' f^* \right) = f'^{\mu\nu} f^*_{\nu\mu}, \qquad \left(\tilde{\varphi} f^{(\prime)} \right) = \tilde{\varphi}^{\mu\nu} f^{(\prime)}_{\nu\mu}$$

Field Induced Part of the Amplitude

- The $\gamma \to \gamma$ amplitude in the fieldless limit

$$\mathcal{M}^{(0)}_{\rm VT} = \frac{eQ_f g_T m_f}{16\pi^2} \, (f' f^*) \int_0^\infty \frac{dt}{t} \int_0^1 du \, e^{-it\left[m_f^2 - q^2\,(1-u^2)/4\right]}$$

- Field-induced part is obtained after subtraction of $\mathcal{M}^{(0)}_{\rm VT}$

$$\Delta\mathcal{M}_{\rm VT} = \mathcal{M}_{\rm VT} - \mathcal{M}^{(0)}_{\rm VT}$$

- The strong field limit, i. e. lowest Landau level contribution

$$\mathcal{M}^{(\rm smf)}_{\rm VT} = \frac{eQ_f g_T m_f \beta q_\perp^2}{8\pi^2 (q_\parallel^2)^2} \, e^{-q_\perp^2/(2\beta)} \, (\tilde{\varphi} f') \, (\tilde{\varphi} f^*) \, F(z)$$

- Introduce $z = 4m_f^2/q_\parallel^2$ and used the function

$$F(z) = \begin{cases} \frac{1}{2\sqrt{1-z}} \left[\ln \left| \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1} \right| - i\pi\Theta(z) \right], & z < 1 \\ \frac{1}{\sqrt{z-1}} \arctan \frac{1}{\sqrt{z-1}}, & z \geq 1 \end{cases}$$

Correlator of Two Tensor Currents

- Tensor-tensor (TT) correlator, $\Pi^{(\mathrm{TT})}_{\mu\nu\rho\sigma}$, is rank-4 tensor
- Antisymmetry of both tensor currents leaves 36 non-trivial coefficients in the basis decomposition
- Of them, eight coefficients only are independent
- Double-integral representation of coefficients is used

$$\Pi^{(\mathrm{TT})}_{ijkl}(q^2, q_\perp^2, \beta) = \frac{1}{4\pi^2} \int_0^\infty \frac{dt}{t} \int_0^1 du\, e^{-i\Omega(t,u)}\, Y^{(\mathrm{TT})}_{ijkl}(q^2, q_\perp^2, \beta; t, u)$$

Integrands in Tensor-Tensor Correlator

- Coefficients relevant for the photon polarization operator

$$Y^{(\mathrm{TT})}_{1414}(q_\parallel^2, q_\perp^2, \beta; t, u) = -q_\perp^2 \left\{ 2q_\perp^2\left(q_\perp^2 + q_\parallel^2\right)\frac{\cos(\beta t) - \cos(\beta t u)}{\sin^2(\beta t)} \right.$$
$$+4q_\perp^2 q_\parallel^2 \left[\cos(\beta t u) - u\sin(\beta t u)\cot(\beta t)\right] - q_\parallel^2\left[\left(1-u^2\right)q_\parallel^2 + 4m_f^2\right]\cos(\beta t u)$$
$$\left. -q_\perp^2\left[\left(1-u^2\right)q_\parallel^2 - 4m_f^2\right]\cos(\beta t) + \frac{4i}{t}q_\parallel^2\left[\cos(\beta t) - \frac{\beta t}{\sin(\beta t)}\right]\right\}$$

$$Y^{(\mathrm{TT})}_{2424}(q_\parallel^2, q_\perp^2, \beta; t, u) = \frac{q_\parallel^2}{q_\perp^2}\, Y^{(\mathrm{TT})}_{1414}(q_\parallel^2, q_\perp^2, \beta; t, u)$$

- Other six coefficients and TT part of the $\gamma \to \gamma$ amplitude will be presented in a forthcoming paper

AMM Contribution to Photon Polarization Operator

- Field-induced part of $\Pi^{(\lambda)}(q)$ is modified (for electrons)

$$\Pi^{(\lambda)}(q) = -i\,\mathcal{P}(q^2) - \frac{\alpha}{\pi}\,Y^{(\lambda)}_{VV} + \frac{\alpha}{\pi}\,a_e\,Y^{(\lambda)}_{VT} + \frac{\alpha}{\pi}\,a_e^2\,Y^{(\lambda)}_{TT}$$

- Last two terms can be presented in the form of double integral

$$Y^{(\lambda)}_{VT(TT)} = \int_0^\infty \frac{dt}{t}\int_0^1 du \left\{ \frac{\beta t}{\sin(\beta t)}\,y^{(\lambda)}_{VT(TT)}\,e^{-i\Omega} - q^2\,e^{-i\Omega_0} \right\}$$

- Notations are from the book by A. Kuznetsov and N. Mikheev
- Part independent on the field is subtracted
- Integrands of vector-tensor part

$$y^{(1)}_{VT} = y^{(3)}_{VT} = q^2\cos(\beta t u)$$
$$y^{(2)}_{VT} = q_{\parallel}^2\cos(\beta t u) - q_{\perp}^2\cos(\beta t)$$

- For the electron, $a_e \sim \alpha$ and the AMM correction is small

AMM Contribution to Photon Polarization Operator

- Integrands of tensor-tensor part

$$y_{TT}^{(1)} = \frac{Y_{1414}^{(TT)}}{4m_e^2\, q_\perp^2}, \qquad y_{TT}^{(2)} = \frac{Y_{2424}^{(TT)}}{4m_e^2\, q_\|^2}$$

- For the electron, tensor-tensor term gives α-suppressed correction to vector-tensor one
- If neutrinos have local interaction with photon due to AMM, they contribute to TT part of photon polarization operator
- Taking into account the upper limit on neutrino AMM $\mu_\nu < 6.4 \times 10^{-12} \mu_B$ [PDG, 2022], this contribution, being $\sim \mu_\nu^2$, is negligible

Conclusions

- Two-point fermionic correlators in presence of constant homogeneous external magnetic field are considered
- This analysis extends the previous one by inclusion the tensor current into consideration
- Study of correlators of tensor fermionic current with the others allows to investigate effects of the fermion anomalous magnetic moment in the one-loop approximation
- Field-induced contribution to the photon polarization operator linear and quadratic in fermion anomalous magnetic moment are calculated
- Computer technique developed for two-point correlators is planned to be applied for three-point ones

Backup Slides

Crossed-Field Limit

- Pure field invariant vanishes: $\beta \to 0$
- Dynamical parameter: $\chi_f^2 = e^2 Q_f^2 (qFFq) = \beta^2 q_\perp^2$
- The crossed-field limit is valid for an ultrarelativistic particle moving in the direction transverse to the field strength in a relatively weak magnetic field, $\chi_f^2 \gg \beta^3$
- Gauge and translationally invariant part of the fermion propagator [A. Kuznetsov & N. Mikheev, Electroweak Processes in External Electromagnetic Fields (Springer, 2013)]

$$S_F(X) = \frac{-i}{32\pi^2} \int_0^\infty \frac{ds}{s^3} \left\{ (X\gamma) - \frac{2\beta^2 s^2}{3} (X\Lambda\gamma) \right.$$
$$\left. + i\beta s \, (X\tilde{\varphi}\gamma)\, \gamma_5 + m_f s \left[2 + \beta s \, (\gamma\varphi\gamma)\right] \right\}$$
$$\times \exp\left\{ -i \left[m_f^2 s + \frac{X^2}{4s} + \frac{\beta^2 s}{12} (X\Lambda X) \right] \right\}$$

- $\varphi_{\mu\nu}$ and $\Lambda_{\mu\nu}$ have the same definitions as in the magnetic field

Crossed-Field Limit

- Pure field invariant vanishes $(\beta \to 0)$
- As basic vectors, accept the following orthonormalized set

$$b_\mu^{(1)} = \frac{eQ_f}{\chi_f}(qF)_\mu, \qquad b_\mu^{(2)} = \frac{eQ_f}{\chi_f}(q\tilde{F})_\mu$$

$$b_\mu^{(3)} = \frac{e^2Q_f^2}{\chi_f^2\sqrt{q^2}}\left[q^2(qFF)_\mu - (qFFq)\,q_\mu\right], \quad b_\mu^{(4)} = \frac{q_\mu}{\sqrt{q^2}}$$

- Dynamical parameter: $\chi_f^2 = e^2Q_f^2(qFFq) = \beta^2 q_\perp^2$
- Coefficients of the vector-tensor correlator in this basis:

$$\Pi_{ijk}^{(VT)}(q^2,\chi_f) = \frac{1}{4\pi^2}\int_0^\infty \frac{dt}{t}\int_0^1 du\, Y_{ijk}^{(VT)}(q^2,\chi_f;t,u)$$

$$\times \exp\left\{-i\left[\left(m_f^2 - \frac{q^2}{4}(1-u^2)\right)t + \frac{1}{48}\chi_f^2(1-u^2)^2t^3\right]\right\}$$

Vector-Tensor Correlator Integrands in Crossed Fields

- Results for integrands in external electromagnetic crossed fields

$$
\begin{aligned}
Y_{114}^{(\mathrm{VT})} &= -Y_{141}^{(\mathrm{VT})} = -m_f\sqrt{q^2} \\
Y_{223}^{(\mathrm{VT})} &= -Y_{232}^{(\mathrm{VT})} = m_f\frac{\chi_f^2 t^2}{2\sqrt{q^2}}\left(1-u^2\right) \\
Y_{224}^{(\mathrm{VT})} &= -Y_{242}^{(\mathrm{VT})} = -m_f\sqrt{q^2}\left[1+\frac{\chi_f^2 t^2}{2q^2}\left(1-u^2\right)\right] \\
Y_{334}^{(\mathrm{VT})} &= -Y_{343}^{(\mathrm{VT})} = -m_f\sqrt{q^2}
\end{aligned}
$$

Integrands of Tensor-Tensor Correlator in Crossed Fields

- Double-integral representation of coefficients is used again

$$\Pi^{(TT)}_{ijk\ell}(q^2,\chi_f) = \frac{1}{4\pi^2}\int_0^\infty \frac{dt}{t}\int_0^1 du\, Y^{(TT)}_{ijk\ell}(q^2,\chi_f;t,u)$$
$$\times \exp\left\{-i\left[\left(m_f^2 - \frac{q^2}{4}\left(1-u^2\right)\right)t + \frac{1}{48}\chi_f^2\left(1-u^2\right)^2 t^3\right]\right\}$$

- Integrands of the tensor-tensor correlator contributing to the photon polarization tensor

$$Y^{(\mathrm{TT})}_{1414} = q^2\left(1-u^2\right) + 4m_f^2 - \frac{t^2\chi_f^2}{12}\left(1-u^2\right)\left(3+5u^2\right)$$
$$+\frac{2m_f^2t^2\chi_f^2}{q^2}\left(1-u^2\right) + \frac{t^4\chi_f^4}{72q^2}\left(1-u^2\right)^2\left(9-u^2\right) + \frac{8it^2\chi_f}{3q^2}$$

$$Y^{(\mathrm{TT})}_{2424} = q^2\left(1-u^2\right) + 4m_f^2 - \frac{t^2\chi_f^2}{12}\left(1-u^2\right)\left(3+5u^2\right)$$
$$+\frac{2m_f^2t^2\chi_f^2}{q^2}\left(1-u^2\right) + \frac{t^4\chi_f^4}{72q^2}\left(1-u^2\right)^2\left(9-u^2\right) + \frac{8it^2\chi_f}{3q^2}$$

- The other coefficients will be presented in a forthcoming paper