
The conference is dedicated to Valery Rubakov memory

# NON-RENORMALIZABLE THEORIES

DMITRY KAZAKOV
BLTP JINR

Motivation:

- The Standard Model is renormalizable
- Gravity is not renormalizable
- Cosmological models are not renormalizable

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

Motivation:

- The Standard Model is renormalizable
- Gravity is not renormalizable
- Cosmological models are not renormalizable

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

- In this talk I consider the first problem and try to answer the question:

How one can obtain unambiguous predictions for the S-matrix in non-renormalizable theories ?

Motivation:

- The Standard Model is renormalizable
- Gravity is not renormalizable
- Cosmological models are not renormalizable

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

- In this talk I consider the first problem and try to answer the question:

How one can obtain unambiguous predictions for the S-matrix in non-renormalizable theories ?

- The answer to the second problem was given in our papers earlier:

To sum up the leading asymptotics in all orders of PT (generalized RG) and to study the high-energy behaviour afterwards

# Effective action and the S-matrix

Effective action

$$e^{i\Gamma(\Phi)} = e^{iS(\Phi)} \int \mathcal{D}\phi e^{i\{S(\Phi+\phi) - S(\Phi) - \frac{\delta\Gamma(\Phi)}{\delta\Phi}\phi\}}, \qquad S(\Phi) = \int d^D x \mathcal{L}(\Phi)$$

$\Phi$ Is a classical background field

Equation of motion
$$\frac{\delta\mathcal{L}(\Phi_{cl})}{\delta\Phi_{cl}} = 0$$

Asymptotic field
$$\phi_{in}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3\epsilon(p)}}\{a(\vec{p})e^{-ipx} + a^*(\vec{p})e^{ipx}\}$$

$$\Phi_{cl}(x) = \phi_{in}(x) + \int dy \Delta_{ret}(x-y)j(y)$$

S-matrix
$$S[a, a^*] = \Gamma[\phi_{in}]$$

$$\Gamma[\Phi_{cl}] = \Gamma[\phi_{in}] + \Delta\Gamma[\phi_{in}] = S[a, a^*] + \Delta S[a, a^*]$$

# Effective action and the S-matrix

Effective action

$$e^{i\Gamma(\Phi)} = e^{iS(\Phi)} \int \mathcal{D}\phi e^{i\{S(\Phi+\phi)-S(\Phi)-\frac{\delta\Gamma(\Phi)}{\delta\Phi}\phi\}}, \qquad S(\Phi) = \int d^D x \mathcal{L}(\Phi)$$

$\Phi$ Is a classical background field

Equation of motion

$$\frac{\delta\mathcal{L}(\Phi_{cl})}{\delta\Phi_{cl}} = 0$$

Asymptotic field

$$\phi_{in}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3\epsilon(p)}}\{a(\vec{p})e^{-ipx} + a^*(\vec{p})e^{ipx}\}$$

$$\Phi_{cl}(x) = \phi_{in}(x) + \int dy \Delta_{ret}(x-y)j(y)$$

S-matrix

$$S[a, a^*] = \Gamma[\phi_{in}]$$

Do not contribute to the S-matrix

$$\Gamma[\Phi_{cl}] = \Gamma[\phi_{in}] + \Delta\Gamma[\phi_{in}] = S[a, a^*] + \Delta S[a, a^*]$$

# Use of equations of motion

Field transformation

$$(*) \qquad \Phi(x) \to \Phi(x) + \Delta\Phi(x)$$

$\Delta\Phi \sim \lambda$ ← Coupling constant

Lagrangian transformation

$$\mathcal{L}[\Phi] \to \mathcal{L}[\Phi] + \mathcal{L}'[\Phi]\Delta\Phi + +\mathcal{O}(\lambda^2)$$

- The S-matrix elements with the proper external lines renormalization factors are not influenced by the replacement of the fields $(*)$
- From this it follows that for any change in $\Delta\mathcal{L}$ which is proportional to $\mathcal{L}'[\Phi]$ does not influence the S-matrix. In other words one can use equations of motion

$$\mathcal{L}'[\Phi^{cl}] = 0$$

to simplify expressions for $\Delta\mathcal{L}$

# Local counter terms on mass shell

Counter terms $\Delta\mathcal{L} = \sum_{i=1}^{N} z_i O_i(\Phi)$

Local operators

Equations of motion $R_j(\Phi) = 0$

Off shell

$$\Delta\mathcal{L} = \sum_{i=1}^{N} \sum_{j=1}^{M=N-K} c_{ij} z_i O_j(\Phi) + \sum_{j=1}^{K} z_j \Phi_j R_j(\Phi)$$

On shell

$$\Delta\tilde{\mathcal{L}} = \sum_{i=1}^{N} \sum_{j=1}^{M=N-K} c_{ij} z_i \tilde{O}_j(\Phi_{cl})$$

# Local counter terms on mass shell

Example: Renormalizable theories $\phi_4^4$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Equation of motion $R_1 = \partial^2\phi + m^2\phi + \lambda/3!\phi^3 = 0$

Off shell $$\Delta\mathcal{L} = -z_1\frac{1}{2}\Phi\partial^2\Phi - \frac{1}{2}z_2 m^2\Phi^2 - z_4\frac{\lambda}{4!}\Phi^4,$$

$$\Delta\mathcal{L} = -\frac{1}{2}(z_2 - z_1)m^2\Phi^2 - (z_4 - 2z_1)\frac{\lambda}{4!}\Phi^4 - \frac{z_1}{2}\Phi(\partial^2\Phi + m^2\Phi + \frac{\lambda}{6}\Phi^3),$$

On shell $$\Delta\tilde{\mathcal{L}} = -\frac{1}{2}(z_2 - z_1)m^2\Phi^2 - (z_4 - 2z_1)\frac{\lambda}{4!}\Phi^4,$$

Renormalization of the couplings $z_m = z_2 - z_1, z_\lambda = z_4 - 2z_1$

# QED

Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial_\mu\gamma^\mu - m + eA_\mu\gamma^\mu)\psi$$

Equations of motion

$$\partial_\mu F_{\mu\nu} + e\bar{\psi}\gamma_\mu\psi = 0,$$
$$(i\partial_\mu\gamma^\mu - m + eA_\mu\gamma^\mu)\psi = 0$$

Counter terms

$$\Delta\mathcal{L} = -z_3\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + z_2\bar{\psi}i\partial_\mu\gamma^\mu\psi - z'm\bar{\psi}\psi + z_1e\bar{\psi}A_\mu\gamma^\mu\psi$$

Off shell

$$\Delta\mathcal{L} = -(z' - z_2)m\bar{\psi}\psi + (z_1 - z_2 - \frac{1}{2}z_3)e\bar{\psi}A_\mu\gamma^\mu\psi$$
$$-z_3\frac{1}{4}(F_{\mu\nu}F_{\mu\nu} + 2e\bar{\psi}\gamma_\mu\psi) + z_2\bar{\psi}(i\partial_\mu\gamma^\mu - m + eA_\mu\gamma^\mu)\psi$$

On shell

$$\Delta\tilde{\mathcal{L}} = -(z' - z_2)m\bar{\psi}\psi + (z_1 - z_2 - \frac{1}{2}z_3)e\bar{\psi}A_\mu\gamma^\mu\psi.$$

Renormalization of the couplings $\quad z_m = z' - z_2, \quad z_e = z_1 - z_2 - \frac{1}{2}z_3$

# Loop Expansion (non-renormalizable case)

$\phi_6^4$

UV divergences within dim reg

$$\Delta\mathcal{L}_1 \sim \lambda^2 (s+t+u)\Phi^4 (\frac{1}{\epsilon} + c_{11})$$

Momentum space

$$\Delta\mathcal{L}_1 \sim \lambda^2 \partial^2 \Phi^2 \Phi^2 (\frac{1}{\epsilon} + c_{11}),$$

Coordinate space

$$\Delta\mathcal{L} = \underset{(\frac{1}{\epsilon}+c_{11})}{\lambda^2\partial^2\Phi^2\Phi^2} + \lambda^3[\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{12})}{\partial^4\Phi^2\Phi^2} + \underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{13})}{\partial^2\Phi^2\partial^2\Phi^2)}] + \lambda^4[...] + \lambda^5[...]$$

$$\searrow \qquad \searrow \qquad \searrow$$

Off shell

$$\underset{(\frac{1}{\epsilon}+c_{21})}{\lambda^3\Phi^6} + \lambda^4[\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{22})}{\partial^2\Phi^4\Phi^2} + \underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{23})}{\partial^2\Phi^2\Phi^4}]$$

$$\searrow \qquad \searrow$$

$$\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{32}),}{\lambda^5\Phi^8}$$

On shell

$$\Delta\tilde{\mathcal{L}} = \lambda^2\partial^2\Phi^2\Phi^2(\frac{1}{\epsilon} + c_{11} + c_{21})$$
$$+\lambda^3[\partial^4\Phi^2\Phi^2(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{12} + c_{22} + c_{32}) + \partial^2\Phi^2\partial^2\Phi^2(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{13} + c_{23})] + ...$$

# The Amplitudes

## Renormalizable case

$\phi^4_4$

$$A_4 = \lambda + A\lambda^2(\frac{3}{\epsilon} - \log(s/\mu) - \log(t/\mu) - \log(u/\mu)) + \ldots$$ One loop

$$\Delta\mathcal{L} = A\frac{\lambda^2}{4!}\Phi^4(-\frac{3}{\epsilon} - c)$$ ← Arbitrary constant

$$A_4^{finite} = \lambda + A\lambda^2(-\log(s/\mu) - \log(t/\mu) - \log(u/\mu) - c) + \ldots$$

$$A_4^0 = \lambda + A\lambda^2(\log(s_0/\mu) - \log(t_0/\mu) - \log(u_0/\mu) - c).$$ Normalization

$$\lambda = A_4^0 + (A_4^0)^2 A(\log(s_0/\mu) - \log(t_0/\mu) - \log(u_0/\mu) - c)$$

$$A_4^{finite} = A_4^0 + (A_4^0)^2 A(-\log(s/s_0) - \log(t/t_0) - \log(u/u_0)) + \ldots$$

- Summary: To fix the arbitrariness it is enough to put one condition on one scattering amplitude. All the rest is calculated unambiguously.

# The Amplitudes

$\phi_6^4$

Non-Renormalizable case

$$A_4 = \lambda + A\lambda^2\left(\frac{s+t+u}{\epsilon} - s\log(s/\mu) - t\log(t/\mu) - u\log(u/\mu)\right) + \dots$$
$$= \lambda + A\lambda^2(-s\log(s/u) - t\log(t/u)) + \dots$$

$$A_6 = B\lambda^3\left(\frac{1}{\epsilon} - \log Q/\mu\right) + \dots$$

$s+t+u=0$

On shell $\Delta\mathcal{L} = \partial^2\Phi^2\Phi^2 A\lambda^2 \times 0 + B\lambda^3\Phi^6\left(-\frac{1}{\epsilon} - c\right)$ Arbitrary constant

$$A_4^{finite} = \lambda + A\lambda^2(-s\log(s/u) - t\log(t/u)),$$
$$A_6^{finite} = B\lambda^3(-\log Q/\mu - c).$$

$$A_4^0 = \lambda + A\lambda^2(-s_0\log(s_0/u_0) - t_0\log(t_0/u_0))$$

$$A_6^0 = B\lambda^3(-\log Q_0/\mu)$$

Normalization

- Summary: to fix the complete arbitrariness in the amplitudes on the mass shell it is necessary to impose an infinite number of conditions on the 4-point amplitude (in the finite order of PT, the number of conditions is finite) and one on the 6-point amplitude.

$\phi_8^4$

# The Amplitudes

$$A_4 = \quad \lambda + A\lambda^2(\frac{s^2+t^2+u^2}{\epsilon} - s^2\log(s/\mu) - t^2\log(t/\mu) - u^2\log(u/\mu)) + ...$$

$$A_6 = B\lambda^3 Q^2(\frac{1}{\epsilon} - \log Q/\mu) + ...$$

$$A_8 = E\lambda^4(\frac{1}{\epsilon} - \log P/\mu) + ...$$

Arbitrary constants

$$\Delta\mathcal{L} = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{1}{\epsilon} - c_1) + B\lambda^3\partial^2\Phi^2\Phi^4(-\frac{1}{\epsilon} - c_2) + E\lambda^4\Phi^8(-\frac{1}{\epsilon} - c_3)$$ Off shell

$$\Delta\tilde{\mathcal{L}} = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{A+B+E}{\epsilon} - c_1 - c_2 - c_3) = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{A+B+E}{\epsilon} - c),$$ On shell

$$A_4^{finite} = \lambda + A\lambda^2(-s^2\log(s/\mu) - t^2\log(t/\mu) - u^2\log(u/\mu) - c(s^2+t^2+u^2)) + ...$$

$$A_6^{finite} = B\lambda^3 Q^2(-\log Q/\mu) + ...,$$

$$A_8^{finite} = E\lambda^4(-\log P/\mu) + ...$$

Now, to fix arbitrariness, we will impose two conditions on the 4-point amplitude and one condition on the 6-point amplitude. The first fixes $\lambda$ and c ($\mu$ will fall out), and the second - $\mu$

$\phi_8^4$

# The Amplitudes

$$A_4 = \lambda + A\lambda^2(\frac{s^2+t^2+u^2}{\epsilon} - s^2\log(s/\mu) - t^2\log(t/\mu) - u^2\log(u/\mu)) + ...$$

$$A_6 = B\lambda^3 Q^2(\frac{1}{\epsilon} - \log Q/\mu) + ...$$

$$A_8 = E\lambda^4(\frac{1}{\epsilon} - \log P/\mu) + ...$$

Arbitrary constants

$$\Delta\mathcal{L} = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{1}{\epsilon} - c_1) + B\lambda^3\partial^2\Phi^2\Phi^4(-\frac{1}{\epsilon} - c_2) ... c_3)$$ Off shell

$$\Delta\tilde{\mathcal{L}} = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{A+B+E}{\epsilon} - c_1 - c_2 - c_3) = ... \frac{... + E}{\epsilon} - c),$$ On shell

$$A_4^{finite} = \lambda + A\lambda^2(-s^2\log(s/ ... - u^2\log(u/\mu) - c(s^2+t^2+u^2)) + ...$$

$$A_6^{finite} = B\lambda^3 Q^2(-1 ...$$

$$A_8^{finite} = ... ) + ...$$

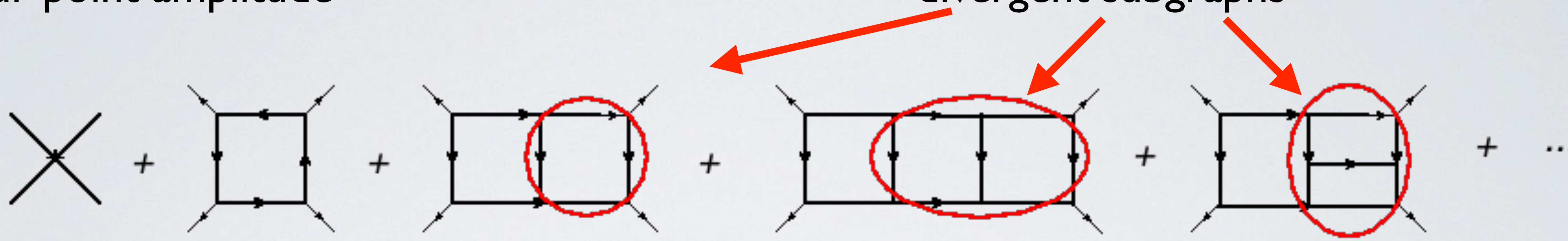
Summary: to fix the complete arbitrariness in the amplitudes on the mass shell it is necessary to impose an infinite number of conditions on the 4-point amplitude (in the finite order of PT, the number of conditions is finite) and one on the 6-point amplitude.

Now, to fix ... iness, we will impose two conditions on the 4-point amplitude and one condition on ... the 6-point amplitude. The first fixes $\lambda$ and c ($\mu$ will fall out), and the second - $\mu$

# Counter-terms on the mass shell

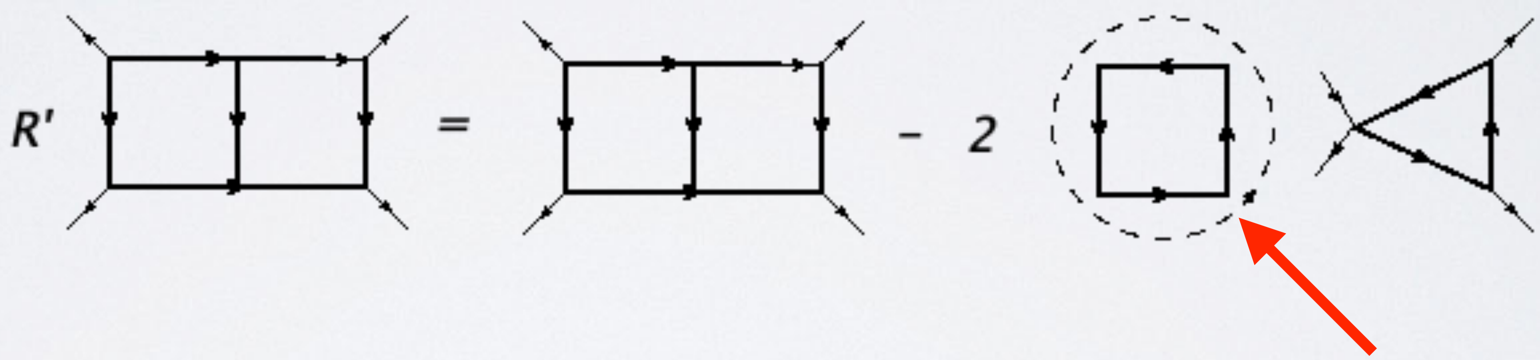
Four point amplitude

divergent subgraphs

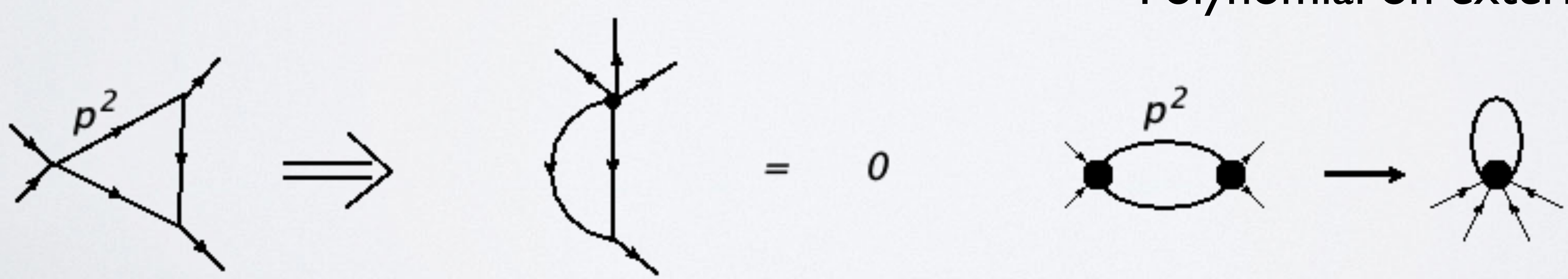



BPHZ R-operation $\Delta\mathcal{L}(G) = -\mathcal{K}\mathcal{R}'G \qquad \mathcal{R}'G = G - \sum_{\gamma} \mathcal{K}\mathcal{R}'_{\gamma} G_{/\gamma} + \sum_{\gamma,\gamma'} \mathcal{K}\mathcal{R}'_{\gamma}\mathcal{K}\mathcal{R}'_{\gamma'} G_{/\gamma\gamma'} - \ldots$

R'-operation




Polynomial on external momenta




Resume: To get the final S-matrix it is enough to have counter terms constructed on mass shell !

# Conclusions

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are <u>unambiguously</u> fixed by the normalization conditions.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are unambiguously fixed by the normalization conditions.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are unambiguously fixed by the normalization conditions.

- The procedure for obtaining predictions for observables - scattering amplitudes on a mass shell - is as follows: One has to calculate the amplitudes directly on the mass shall, subtract all divergences with arbitrary subtraction constants and then fix these constants by imposing conditions on the scattering amplitudes.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are unambiguously fixed by the normalization conditions.

- The procedure for obtaining predictions for observables - scattering amplitudes on a mass shell - is as follows: One has to calculate the amplitudes directly on the mass shall, subtract all divergences with arbitrary subtraction constants and then fix these constants by imposing conditions on the scattering amplitudes.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are unambiguously fixed by the normalization conditions.

- The procedure for obtaining predictions for observables - scattering amplitudes on a mass shell - is as follows: One has to calculate the amplitudes directly on the mass shall, subtract all divergences with arbitrary subtraction constants and then fix these constants by imposing conditions on the scattering amplitudes.

- In any theory, it is possible to fix the arbitrariness normalizing the amplitude of 2->2 scattering (plus one condition on 3->3 scattering). Then all other amplitudes will be determined unambiguously.

# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.
- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are unambiguously fixed by the normalization conditions.
- The procedure for obtaining predictions for observables - scattering amplitudes on a mass shell - is as follows: One has to calculate the amplitudes directly on the mass shall, subtract all divergences with arbitrary subtraction constants and then fix these constants by imposing conditions on the scattering amplitudes.
- In any theory, it is possible to fix the arbitrariness normalizing the amplitude of 2->2 scattering (plus one condition on 3->3 scattering). Then all other amplitudes will be determined unambiguously.

# Notes

# Notes

- In renormalizable theories the choice of normalization conditions corresponds to the choice of a subtraction scheme. While switching from scheme to scheme the renormalized coupling constants are multiplied by a constant multiplier Z

# Notes

- In renormalizable theories the choice of normalization conditions corresponds to the choice of a subtraction scheme. While switching from scheme to scheme the renormalized coupling constants are multiplied by a constant multiplier Z

# Notes

- In renormalizable theories the choice of normalization conditions corresponds to the choice of a subtraction scheme. While switching from scheme to scheme the renormalized coupling constants are multiplied by a constant multiplier Z

- In non-renormalized theories, renormalization is not multiplicative, but depends on kinematics and has an integral character. The unambiguity of the answer, as in renormalizable theories, is achieved due to the fact that while expressing the multi-leg amplitude through a four-point one with fixed kinematics, the entire dependence on the arbitrariness of subtraction is cancelled.

# Notes

- In renormalizable theories the choice of normalization conditions corresponds to the choice of a subtraction scheme. While switching from scheme to scheme the renormalized coupling constants are multiplied by a constant multiplier Z

- In non-renormalized theories, renormalization is not multiplicative, but depends on kinematics and has an integral character. The unambiguity of the answer, as in renormalizable theories, is achieved due to the fact that while expressing the multi-leg amplitude through a four-point one with fixed kinematics, the entire dependence on the arbitrariness of subtraction is cancelled.

# Notes

- In renormalizable theories the choice of normalization conditions corresponds to the choice of a subtraction scheme. While switching from scheme to scheme the renormalized coupling constants are multiplied by a constant multiplier Z

- In non-renormalized theories, renormalization is not multiplicative, but depends on kinematics and has an integral character. The unambiguity of the answer, as in renormalizable theories, is achieved due to the fact that while expressing the multi-leg amplitude through a four-point one with fixed kinematics, the entire dependence on the arbitrariness of subtraction is cancelled.

- In non-renormalized theories, the amplitudes grow with energy in each fixed order of perturbation theory

$$g^n s^n \log s$$

and one has to sum up the leading asymptotics in all orders of PT (generalized RG) and to study the high-energy behaviour afterwards