Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections

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Geometrical formalism

Affine connection:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}\,, \tag{1}$$

Nonmetricity tensor:

$$Q_{\rho\mu\nu} \equiv \nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \tilde{\Gamma}^{\beta}{}_{\mu\rho} g_{\beta\nu} - \tilde{\Gamma}^{\beta}{}_{\nu\rho} g_{\mu\beta}\,. \tag{2}$$

Disformation tensor:

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}\right) = L^{\lambda}{}_{\nu\mu}\,. \tag{3}$$

Relation between GR and STEGR

$$\mathring{R} = Q + \mathring{\nabla}_{\mu}(\hat{Q}^{\mu} - Q^{\mu})\,.$$

where the nonmetricity scalar and traces are defined as [1]

$$Q \equiv -\frac{1}{4} Q_{\lambda\mu\nu} Q^{\lambda\mu\nu} + \frac{1}{2} Q_{\lambda\mu\nu} Q^{\mu\nu\lambda} + \frac{1}{4} Q_{\mu} Q^{\mu} - \frac{1}{2} Q_{\mu} \hat{Q}^{\mu}\,, \tag{4}$$

$$Q_{\mu} \equiv Q_{\mu\nu}{}^{\nu}\,, \qquad \hat{Q}_{\mu} \equiv Q_{\nu\mu}{}^{\nu}\,. \tag{5}$$

$$P^{\alpha}{}_{\mu\nu} = \frac{1}{2} \frac{\partial Q}{\partial Q_{\alpha}{}^{\mu\nu}}\,. \tag{6}$$

[1] Note that some authors define Q with the opposite overall sign,

Scalar non-metricity gravity

Scalar non-metricity action:

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left(\mathcal{A}(\Phi)Q - \mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - 2\mathcal{V}(\Phi)\right) + S_{\mathrm{m}}\,, \tag{7}$$

varying w.r.t. metric:

$$\mathcal{A}(\Phi)\mathring{G}_{\mu\nu} + 2\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Phi}P^{\lambda}{}_{\mu\nu}\partial_\lambda\Phi + \frac{1}{2}g_{\mu\nu}\left(\mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + 2\mathcal{V}(\Phi)\right) - \mathcal{B}(\Phi)\partial_\mu\Phi\partial_\nu\Phi = \kappa^2\mathcal{T}_{\mu\nu}\,, \tag{8}$$

varying w.r.t. connection:

$$\left(\frac{1}{2}Q_\beta + \nabla_\beta\right)\left[\partial_\alpha\mathcal{A}\left(\frac{1}{2}Q_\mu g^{\alpha\beta} - \frac{1}{2}\delta^\alpha_\mu Q^\beta - Q_\mu{}^{\alpha\beta} + \delta^\alpha_\mu Q_\gamma{}^{\gamma\beta}\right)\right] = 0, \tag{9}$$

varying w.r.t. scalar field:

$$2\mathcal{B}\mathring{\Box}\Phi + \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\Phi}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Phi}Q - 2\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\Phi} \quad = \quad 0\,. \tag{10}$$

The continuity equation of the matter fields [2]

$$\mathring{\nabla}_\mu\mathcal{T}^\mu{}_\nu = 0.$$

[2] L. Järv, M. Rünkla, M. Saal, and O. Vilson, Phys. Rev. D, 97 (2018) no. 12, 124025, DOI: https://doi.org/10.1103/PhysRevD.97.124025.

Spatially homogeneous and isotropic:

$$ds^2 = -dt^2 + a(t)^2 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi \right) \tag{11}$$

Branch of connection

Spatially flat FLRW symmetry is satisfied by three sets of connections, introducing an extra function γ. [3]

$$\Gamma^{\rho}{}_{\mu\nu} = \left[\begin{bmatrix} \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right] \tag{12}$$

$$\Gamma^{\rho}{}_{\mu\nu} = \left[\begin{bmatrix} \gamma(t) + \frac{\dot\gamma(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma(t) & 0 & 0 \\ \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & \gamma(t) & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \gamma(t) & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \gamma(t) \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \gamma(t) & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right] \tag{13}$$

$$\Gamma^{\rho}{}_{\mu\nu} = \left[\begin{bmatrix} -\frac{\dot\gamma(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & \gamma(t) & 0 & 0 \\ 0 & 0 & r^2\gamma(t) & 0 \\ 0 & 0 & 0 & r^2\gamma(t)\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right] \tag{14}$$

[3] M. Hohmann, Phys. Rev. D 104 (2021) 12, DOI: https://doi.org/10.1103/PhysRevD.104.124077

Objective

• Under which conditions these cosmological spacetime configurations with radiation, dust matter, and potential content relax to the limit of general relativity where the variation of the gravitational constant ceases and the system evolves close to general relativity?

• Does the extra free function γ in the connection bring a new degree of freedom?

• Does this extra function play any role in the context of dark matter or dark energy?

• Our analysis encompassed all possible cases to investigate the stability of the solution associated with each branch in the radiation-dominated, dust matter, and potential-dominated eras.

Metric equation:

$$6H^2\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho\,, \tag{15}$$

$$-4H\dot{\Phi}\mathcal{A}'(\Phi) - \left(6H^2 + 4\dot{H}\right)\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 \mathrm{w}\rho\,. \tag{16}$$

Scalar field equation:

$$-6H^2\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0 \tag{17}$$

Non-metricity scalar:

$$Q = -6H^2 \tag{18}$$

Compare with GR equation:

$$8\pi G_N = \frac{\kappa^2}{\mathcal{A}(\Phi_*)}, \qquad \Lambda = \frac{\mathcal{V}(\Phi_*)}{\mathcal{A}(\Phi_*)}$$

Metric equation:

$$3\dot{\Phi}\gamma\mathcal{A}'(\Phi) + 6H^2\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho\,, \tag{19}$$

$$(3\dot{\Phi}\gamma - 4H\dot{\Phi})\mathcal{A}'(\Phi) - \left(6H^2 + 4\dot{H}\right)\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2\mathrm{w}\rho\,. \tag{20}$$

Connection equation:

$$3\gamma\left(\ddot{\Phi}\mathcal{A}'(\Phi) + 3H\dot{\Phi}\mathcal{A}'(\Phi) + \dot{\Phi}^2\mathcal{A}''(\Phi)\right) = 0\,, \tag{21}$$

Scalar field equation:

$$(-6H^2 + 9H\gamma + 3\dot{\gamma})\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0 \tag{22}$$

Non-metricity scalar:

$$Q = -6H^2 + 9H\bar{\gamma} + 3\dot{\bar{\gamma}} \tag{23}$$

The connection equation does not provide dynamics for the independent connection function γ but rather restrains the scalar field dynamics to:

$$\ddot{\Phi} = -3H\dot{\Phi} - \frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}\dot{\Phi}^2\,. \tag{24}$$

Connection set III

Metric equation

$$6H^2\mathcal{A}(\Phi) - 3\bar{\gamma}\dot{\Phi}\mathcal{A}'(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho \tag{25}$$

$$(\bar{\gamma}\,\dot{\Phi} - 4H\dot{\Phi})\mathcal{A}'(\Phi) - \left(6H^2 + 4\dot{H}\right)\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 \mathrm{w}\rho \tag{26}$$

Connection equation

$$-6\dot{\bar{\gamma}}\,\dot{\Phi}\mathcal{A}'(\Phi) - 3\bar{\gamma}\left(\ddot{\Phi}\mathcal{A}'(\Phi) + 5H\,\dot{\Phi}\,\mathcal{A}'(\Phi) + \dot{\Phi}^2\mathcal{A}''(\Phi)\right) = 0 \tag{27}$$

Scalar field equation:

$$(-6H^2 + 9H\bar{\gamma} + 3\dot{\bar{\gamma}})\mathcal{A}'(\Phi) - (6H\,\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0 \tag{28}$$

Non-metricity scalar:

$$Q = -6H^2 + 9H\bar{\gamma} + 3\dot{\bar{\gamma}} \tag{29}$$

$$\ddot{\Phi} = -\left(\frac{2\dot{\bar{\gamma}}}{\bar{\gamma}} + \frac{5}{3\bar{\gamma}}H\right)\dot{\Phi} - \frac{1}{3\bar{\gamma}}\frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}\dot{\Phi}^2 \tag{30}$$

Expand around GR

Adopt a parametrization:

$$\mathcal{A}(\phi) = 1 + f(\phi)\,, \quad \mathcal{B}(\phi) = 1 \quad \mathcal{V}(\phi) = V(\phi) \tag{31}$$

Evolution of small perturbation around GR limit:

$$\phi(t) = \phi_* + x(t)\,, \qquad H(t) = H_*(t) + h(t)\,, \qquad \gamma(t) = \gamma_*(t) + g(t)\,, \qquad \rho(t) = \rho_*(t) + r(t)\,. \tag{32}$$

In matter dominated case:

$$H_*(t) = \frac{2}{3(t - t_s)}\,, \qquad \rho_*(t) = \frac{4(1 + f_*)}{3\kappa^2(t - t_s)^2}\,, \qquad V_* = 0\,, \tag{33}$$

In relativistic matter:

$$H_*(t) = \frac{1}{2(t - t_s)}\,, \qquad \rho_*(t) = \frac{3(1 + f_*)}{4\kappa^2(t - t_s)^2}\,, \qquad V_* = 0\,, \tag{34}$$

In potential dominated case:

$$H_*(t) = \sqrt{\frac{V_*}{3(1 + f_*)}}\,, \qquad \rho_*(t) = 0\,, \qquad V_* = \mathrm{const.} \neq 0\,. \tag{35}$$

Note: $\dot{\phi}\gamma = \dot{x}(t)\dot{\gamma}_*(t)$

From the first order small scalar field equation (6) of connection set I:

$$\ddot{x} = -3H_*\dot{x}(t) - 3H_*^2 f_*'' x(t) - V_*'' x(t) \tag{36}$$

The background scalar field equation for connection sets II and III are:

$$\left(3\gamma_*'(t) + 9H_*(t)\gamma_*(t) - 6H_*^2\right) f_*' - 2V_*' = 0. \tag{37}$$

Connection set I

Cases	Matter-Domination	Radiation-Domination	Potential-Domination
$f'_* = 0, V'_* = 0,$ $V''_* = 0$ v_* is imaginary	$x(t) \sim t^{-\frac{1}{4}}, h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$x(t) \sim t^{-\frac{1}{2}}, h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$h(t) = r(t) \sim e^{-3H_* t},$ $x(t) \sim e^{\frac{-3H_* t}{2}}$
$f'_* = 0, V'_* = 0,$ $V''_* = 0$ v_* is real	$x(t) \sim t^{-\frac{1}{2}(1-v_*)},$ $h(t) \sim t^{-2}, r(t) \sim t^{-3}$	$x(t) \sim t^{-\frac{1}{4}(1-v_*)},$ $h(t) \sim t^{-2}, r(t) \sim t^{-3}$	$h(t) = r(t) \sim e^{-3H_* t},$ $x(t) \sim e^{\frac{-3H_* t}{2}(1-v_*)}$
$f'_* = 0, V'_* = 0,$ $V''_* < 0$	$x(t) \sim t^{-1} e^t, h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$x(t) \sim t^{\frac{-3}{4}} e^t, h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$h(t) = r(t) \sim e^{-3H_* t},$ $x(t) \sim e^{\frac{-3H_* t}{2}(1-v_*)}$
$f'_* \neq 0, V'_* = 0,$ $V''_* > 0$	$x(t) \sim t^{-1}, h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$x(t) \sim t^{\frac{-3}{4}}, h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$h(t) = r(t) \sim e^{-3H_* t},$ $x(t) \sim e^{\frac{-3H_* t}{2}(1-v_*)}$

Balanced case

Scalar field at a value which satisfies $V_* f'_* = -V'_*(1+f_*)$. Then the solution from the leading order [4] :

$$r(t) \sim e^{-3H_* t} \tag{38}$$

From the remaining perturbed equations evolve as:

$$h(t) \sim x(t) \sim \begin{cases} e^{-\frac{3H_* t}{2}}, & v_* \text{ imaginary} \\ e^{-\frac{3H_* t}{2}(1-v_*)}, & v_* \text{ real} \end{cases} \quad , \quad v_* = \sqrt{1 - \frac{4f''_*}{3} - \frac{4(1+f_*)V''_*}{3V_*} - \frac{8f'_* V'_*}{V_*}} \tag{39}$$

This configuration is stable if

$$V_* f''_* + (1+f_*)V'' + 2f'_* V'_* > 0$$

[4] Laur Järv and Alexey Toporensky, Phys. Rev. D 93 (2016) 024051, DOI:https://doi.org/10.1103/PhysRevD.93.024051

Connection set II

Cases	Matter-Domination	Radiation-Domination	Potential-Domination
$f'_* \neq 0, V'_* \neq 0$	$\gamma_*(t) \sim t, x(t) \sim t^{-1}$, $h(t) \sim t^0, g(t) \sim t^2$, $r(t) \sim t^{-1}$	$\gamma_*(t) \sim t, x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{\frac{1}{2}}, g(t) \sim t^{\frac{5}{2}}$, $r(t) \sim t^{-\frac{1}{2}}$	$\gamma_*(t) = x(t) \sim e^{-3H_* t}$, $h(t) = r(t) \sim e^{-3H_* t}$, $g(t) \sim te^{-3H_* t}$
$f'_* \neq 0, V'_* = 0$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-1}$, $h(t) \sim t^{-2}, g(t) \sim t^0$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{-2} \ln t, g(t) \sim t^{\frac{1}{2}}$, $r(t) \sim t^{-3} \ln t$	$\gamma_*(t) = x(t) \sim e^{-3H_* t}$, $h(t) = r(t) \sim e^{-3H_* t}$, $g(t) \sim te^{-3H_* t}$
$f'_* \neq 0, V_* =$ Const	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-1}$, $h(t) \sim t^{-2}, g(t) \sim t^{-2}$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{-2} \ln t, g(t) \sim t^{-\frac{3}{2}}$, $r(t) \sim t^{-3} \ln t$	$h(t) = r(t) \sim e^{-3H_* t}$, $x(t) \sim e^{\frac{-3H_* t}{2}}$, $\gamma_*(t) \sim e^{-3H_* t}$, $g(t) \sim te^{-3H_* t}$
$f'_* = 0, V'_* = 0$	$\gamma_*(t) \sim t, \ x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{-2}, g(t) \sim t^{\frac{1}{2}}$, $r(t) \sim t^{-3}$	$h(t) \sim t^{-2}, r(t) \sim t^{-3}$, $x(t) \sim t^{\frac{-1}{4}}, \gamma_*(t) \sim t$, $g(t) \sim t^{\frac{3}{4}}$	$h(t) = r(t) \sim e^{-3H_* t}$, $x(t) \sim e^{\frac{-3H_* t}{2}}$, $\gamma_*(t) \sim e^{-3H_* t}$, $g(t) \sim te^{-3H_* t}$
$f'_* = 0, V_* =$Const	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{-2}, g(t) \sim t^{-\frac{3}{2}}$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^{-1}, h(t) \sim t^{-2}$, $r(t) \sim t^{-3}, x(t) \sim t^{\frac{-1}{4}}$, $g(t) \sim t^0$	$h(t) = r(t) \sim e^{-3H_* t}$, $x(t) \sim e^{\frac{-3H_* t}{2}}$, $\gamma_*(t) \sim e^{-3H_* t}$, $g(t) \sim te^{-3H_* t}$

Sudden singularity

In the case $f'_* = 0$ and $V'_* = 0$, the leading order solution from the connection equation

$$x(t) = \pm\sqrt{\frac{c_6}{t} + c_7}\,, \quad \dot{x}(t) = \mp\frac{c_6}{2t^2\sqrt{\frac{c_6}{t} + c_7}}.$$

$$c_6 = -2\dot{x}_0\, x_0\, t_0^2\,, \qquad c_7 = x_0^2 + 2\dot{x}_0\, x_0\, t_0\,, \tag{40}$$

At finite time $t_* = -\frac{c_6}{c_7}$. If c_6 and c_7 have opposite signs.

$$\frac{t_0}{t_*} = -\frac{c_7\, t_0}{c_6} = 1 + \frac{x_0}{2\dot{x}_0\, t_0} \tag{41}$$

Case-1: If x_0 and $\dot{x}_0$ are of the same sign, then $t_* < t_0$.

Case-2: If x_0 and $\dot{x}_0$ are of opposite signs and $|\dot{x}_0| > \frac{|x_0|}{2t_0}$, then $t_* > t_0$.

Case-3: If x_0 and $\dot{x}_0$ are of opposite signs, c_6 and c_7 are of same sign, then $|\dot{x}_0| < \frac{|x_0|}{2t_0}$

Finally, $\boxed{x_0 = -2\dot{x}_0\, t_0}$, when $c_7 = 0$.

Phase space diagram

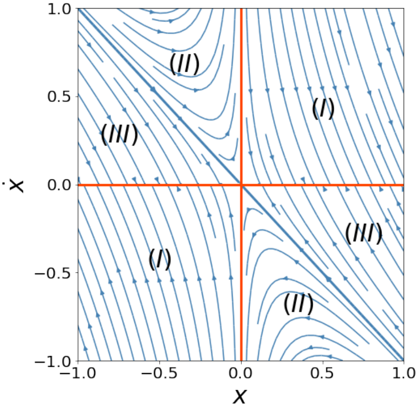

Figure: A sketch of the phase space where singular solutions in occur at $t = 1$

Connection set-III

Cases	Matter-Domination	Radiation-Domination	Potential-Domination
$f'_* \neq 0,\, V'_* \neq 0$	$\gamma_*(t) \sim t, x(t) \sim t^{\frac{-13}{3}}$, $h(t) \sim t^{-2}, g(t) \sim t^0$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t, x(t) \sim t^{-\frac{7}{2}}$, $h(t) \sim t^{-2}, g(t) \sim t^0$, $r(t) \sim t^{-3}$	$\gamma_*(t) = h(t) \sim e^{-3H_* t}$, $x(t) \sim e^{-5H_* t}$, $g(t) \sim te^{-3H_* t}$, $r(t) \sim e^{-3H_* t}$
$f'_* \neq 0,\, V'_* = 0$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{\frac{-1}{3}}$, $h(t) \sim t^{\frac{-4}{3}}, g(t) \sim t^{\frac{2}{3}}$, $r(t) \sim t^{\frac{-7}{3}}$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{\frac{1}{2}}$, $h(t) \sim t^{-2} \ln t, g(t) \sim t^{\frac{3}{2}}$, $r(t) \sim t^{-3}$	$\gamma_*(t) = h(t) \sim e^{-3H_* t}$, $x(t) \sim e^{-5H_* t}$, $g(t) \sim te^{-3H_* t}$, $r(t) \sim e^{-3H_* t}$
$f'_* \neq 0,\, V'_* = V''_* = 0$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{\frac{-1}{3}}$, $h(t) \sim t^{\frac{-4}{3}}, g(t) \sim t^{\frac{-4}{3}}$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{\frac{1}{2}}$, $h(t) \sim t^{-2} \ln t, g(t) \sim t^{-\frac{1}{2}}$, $r(t) \sim t^{-3}$	$\gamma_*(t) = h(t) \sim e^{-3H_* t}$, $x(t) \sim e^{-5H_* t}$, $g(t) \sim te^{-3H_* t}$, $r(t) \sim e^{-3H_* t}$
$f'_* = 0,\, V'_* = 0$	$h(t) \sim t^{-2}, r(t) \sim t^{-3}$ Observed singularity	$h(t) \sim t^{-2}, r(t) \sim t^{-3}$ Observed singularity	$h(t) \sim e^{-3H_* t}, r(t) \sim e^{-3H_* t}$ Observed singularity

Matter dominated $f'_* = 0$ and $V'_* = 0$

From second-order small connection equation

$$\bar{\gamma}_* = \frac{c_9}{t^{\frac{5}{3}}\sqrt{|x||\dot{x}|}}\,. \tag{42}$$

Substitute (42) into the 1st order small scalar field

$$\ddot{x} = -\frac{\dot{x}\left(3c_9 f''_*\left(3t\dot{x} - 2x\right) + 4t^{\frac{2}{3}}\left(3V''_* t^2 x + 4f''_* x + 6t\dot{x}\right)\sqrt{|x|\,|\dot{x}|}\right)}{3t\left(3c_9 f''_* x + 4t^{\frac{5}{3}}\dot{x}\sqrt{|x|\,|\dot{x}|}\right)}\,. \tag{43}$$

In the limit $x \to 0$ we can expand (43) to get

$$\ddot{x} = -\frac{3c_9 f''_*\sqrt{|\dot{x}|}}{4t^{\frac{5}{3}}\sqrt{x}} + \mathcal{O}(x^0)\,, \tag{44}$$

In the limit $\dot{x} \to 0$ we get from (43)

$$\ddot{x} = \frac{2\dot{x}}{3t} + \mathcal{O}(\dot{x}^{\frac{3}{2}})\,, \tag{45}$$

Finally, we notice that the expression (43) has also another string of singularities at

$$x_s = -\frac{16t^{\frac{10}{3}}\dot{x}^3}{9c_9^2(f''_*)^2}\operatorname{sign}(c_9 f''_*) \tag{46}$$

Radiation dominated $f'_* = 0$ and $V'_* = 0$

$$\bar{\gamma}_* = \frac{c_9}{t^{\frac{5}{4}}\sqrt{|x||\dot{x}|}} \,. \tag{47}$$

Substituting that into the first order scalar field equation (42) yields again a highly nonlinear equation

$$\ddot{x}(t) = -\frac{\dot{x}\left(3c_9 f''_*(2t\dot{x} - x) + 2t^{\frac{1}{4}}(3f''_* x + 4V''_* t^2 x + 6t\dot{x})\sqrt{|x||\dot{x}|}\right)}{2t\left(3c_9 f''_* + 4t^{\frac{5}{4}}\dot{x}\sqrt{|x||\dot{x}|}\right)} \,. \tag{48}$$

• In summary, we see that the radiation dominated regime is unstable, even the case of $f'_* \neq 0$ but $V'_* = V''_* = 0$. Again, if a model allows a value ϕ_* where simultaneously $f'_* = 0$ and $V'_* = 0$, then a large class of solutions will likely face a singularity in finite time.

Potential dominated $f'_* = 0$ and $V'_* = 0$

$$\gamma_*(t) = \frac{c_9}{e^{\frac{5H_* t}{2}}\sqrt{|x||\dot{x}|}} \tag{49}$$

$$\ddot{x}(t) = -\frac{\dot{x}\left(3c_9 f''_*(\dot{x} - H_* x) + 4e^{\frac{5H_* t}{2}}(3f''_* H_*^2 x + V''_* x + 3H_*\dot{x})\sqrt{|x||\dot{x}|}\right)}{3c_9 f''_* x + 4e^{\frac{5H_* t}{2}}\dot{x}\sqrt{|x||\dot{x}|}} \,. \tag{50}$$

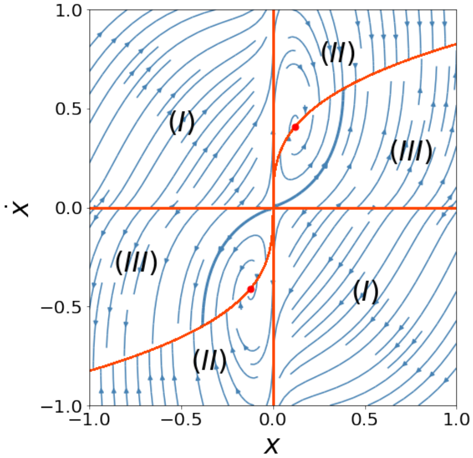

Figure: set 3 dust dominated for $V_*'' = 1$, $f_*'' = 1$, $c_9 = 0.5$ at $t = 1$

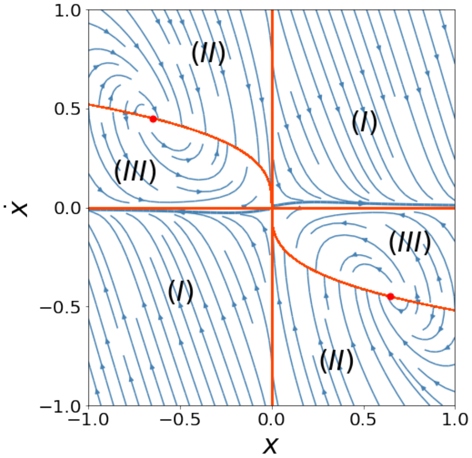

Figure: set 3 dust dominated for $V_*'' = 1$, $f_*'' = 1$, $c_9 = -1$ at $t = 1$.

Results

- In connection set I the extra function γ does not appear in the field equations and is left undetermined. In connection set II and III, it adds a new degree of freedom and affects the dynamics of the system.

- Comparing the metric equations, the extra function γ would have $\mathrm{w} = 1$ in set II and $\mathrm{w} = -\frac{1}{3}$ in set III, and thus can not play the role of dark energy or dark matter.

- In addition we also found that the extra function drastically modifies the scalar field dynamics, since the connection equation can be viewed as a dynamical equation for the scalar field, albeit without any contribution from the kinetic coupling $\mathcal{B}(\Phi)$ or scalar potential $\mathcal{V}(\Phi)$, which look quite puzzling.

- We determined the restrictions on the model functions which permit the standard cosmological scenario of successive radiation, dust matter, and scalar potential domination eras to be stable. However, the alternative connections also introduce a rather general possibility of the system meeting a singularity in finite time.

Results

Branch	Matter-Domination	Radiation-Domination	Potential-Domination
Connection set I	$f'_* = 0,\ V'_* = 0,\ V''_* > 0$	$f'_* = 0,\ V'_* = 0,\ V''_* > 0$	$f'_* = 0,\ V'_* = 0,\ V''_* > 0$
	$f'_* = 0,\ f''_* > 0,\ V'_* = 0$	$f'_* = 0,\ f''_* > 0,\ V'_* = 0$	$f'_* = 0,\ f''_* > 0,\ V'_* = 0$
Connection set II	$f'_* \neq 0,\ V_* =$ Constant	$f'_* \neq 0,\ V_* =$ Constant	$f'_* \neq 0,\ V_* =$ Constant
Connection set III	$f'_* \neq 0$ and $V'_* = V''_* = 0$	No stable scenario	$f'_* \neq 0$

- In conclusion, the alternative FLRW connections can not be deemed outright pathological and do not make the universe definitely unstable, but they have a very strange and possibly dangerous influence on the scalar field dynamics nevertheless.

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Thanks for your attention!