Asymptotic freedom in (3+1)-dimensional projectable Hořava gravity

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Motivation for Hořava gravity

Einstein GR

$$S_{EH} = \frac{M_P^2}{2} \int dt\, d^d x\, \sqrt{-g} R \quad \Rightarrow \quad \frac{M_P^2}{2} \int dt\, d^d x\, \left(h_{ij} \Box h^{ij} + \dots \right) \tag{1}$$

Higher derivative gravity (Stelle 1977)

$$\int \left(R + R^2 + R_{\mu\nu} R^{\mu\nu} \right) \quad \Rightarrow \quad \int \left(h_{ij} \Box h^{ij} + h_{ij} \Box^2 h^{ij} + \dots \right) \tag{2}$$

The theory is renormalizable and asymptotically free. However the theory is not unitary due to presence of ghosts.

Hořava gravity (2009)

The key is the anisotropic scaling of time and space coordinates,

$$t \mapsto b^{-z} t \,, \quad x^i \mapsto b^{-1} x^i \,, \qquad i = 1, \ldots, d \tag{3}$$

The theory contains only second time derivatives

$$\int \underbrace{dt\, d^d x}_{\propto\, b^{-(z+d)}} \Big(\dot{h}_{ij} \dot{h}_{ij} - h_{ij} (-\Delta)^z h_{ij} + \ldots \Big) \tag{4}$$

And field scales as

$$h_{ij} \mapsto b^{(d-z)/2} h_{ij} \tag{5}$$

Critical theory

$$z = d \tag{6}$$

Foliation preserving diffeomorphisms

$$t \mapsto t'(t) \ , \qquad x^i \mapsto x'^i(t, \mathbf{x}) \tag{7}$$

Metric decomposition

The metric in the action of HG is expanded into the lapse N, the shift N^i and the spatial metric γ_{ij} like in the Arnowitt–Deser–Misner (ADM) decomposition,

$$\mathrm{d}s^2 = N^2\mathrm{d}t^2 - \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)\,. \tag{8}$$

Fields are assigned the following dimensions under the anisotropic scaling:

$$[N] = [\gamma_{ij}] = 0\ , \qquad [N^i] = d - 1\ . \tag{9}$$

Projectable version

A.Barvinsky, D.Blas, M.Herrero-Valea, S.Sibiryakov, C.Steinwachs (2016)

We consider *projectable* version of Hořava gravity. The lapse N is restricted to be a function of time only, $N = N(t)$

$$S = \frac{1}{2G} \int \mathrm{d}t \, \mathrm{d}^d x \sqrt{\gamma} \big(K_{ij} K^{ij} - \lambda K^2 - \mathcal{V} \big) \, , \tag{10}$$

where

$$K_{ij} = \frac{1}{2} \left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \, . \tag{11}$$

The potential part $\mathcal{V}$ in $d = 3$ reads,

$$\begin{aligned} \mathcal{V} =& 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij} R^{ij} \\ & + \nu_1 R^3 + \nu_2 R R_{ij} R^{ij} + \nu_3 R^i_j R^j_k R^k_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk} \, , \end{aligned} \tag{12}$$

This expression includes all relevant and marginal terms. It contains 9 couplings $\Lambda, \eta, \mu_1, \mu_2$ and $\nu_a, a = 1, \ldots, 5$.

Dispersion relations

The spectrum of perturbations contains a transverse-traceless graviton and a scalar mode. Both modes have positive kinetic terms when G is positive and

$$\lambda < 1/3 \quad \text{or} \quad \lambda > 1 \, . \tag{13}$$

Their dispersion relations around a flat background are

$$\omega_{tt}^2 = \eta k^2 + \mu_2 k^4 + \nu_5 k^6 \, , \tag{14a}$$

$$\omega_s^2 = \frac{1-\lambda}{1-3\lambda}\big(- \eta k^2 + (8\mu_1 + 3\mu_2)k^4\big) + \nu_s k^6 \, , \tag{14b}$$

where k is the spatial momentum and we have defined

$$\nu_s \equiv \frac{(1-\lambda)(8\nu_4 + 3\nu_5)}{1-3\lambda} \, . \tag{15}$$

These dispersion relations are problematic at low energies where they are dominated by the k^2-terms.

One-loop action

Background field method with static background metric and zero background shift

$$\gamma_{ij}(\tau, \mathbf{x}) = g_{ij}(\mathbf{x}) + h_{ij}(\tau, \mathbf{x}), \quad N^i(\tau, \mathbf{x}) = 0 + n^i(\tau, \mathbf{x})\,. \tag{16}$$

The one-loop effective action is given by the Gaussian path integral

$$\exp\left(-\varGamma^{1-\mathrm{loop}}\right) = \sqrt{\mathrm{Det}\,\mathcal{O}_{ij}} \int \left[\, dh^A\, dn^i\, dc^i\, d\bar{c}_j \right]\, \exp\left(\, - S^{(2)}[\, h^A, n^i, c^i, c_j\,]\,\right),$$

where the quadratic part of the full action consists of three contributions — metric, shift vector and ghost ones,

$$\begin{aligned} S^{(2)}[\, h^A, n^i, c^i, \bar{c}_j\,] = \frac{1}{G} \int d\tau d^3x \sqrt{g} \Big[\frac{1}{2} h^A \left(-\mathbb{G}_{AB}\partial_\tau^2 + \mathbb{D}_{AB}\right) h^B \\ + \frac{1}{2}\sigma n^i \mathcal{O}_{ik} \left(-\delta^k_j \partial_\tau^2 + \mathbb{B}^k{}_j\right) n^j + \bar{c}_i \left(-\delta^i_j \partial_\tau^2 + \mathbb{B}^i{}_j\right) c^j \Big]. \end{aligned} \tag{17}$$

Metric part of the action

The kinetic part for the metric perturbations has the form

$$-\frac{\sqrt{g}}{2G}\, h^A \mathbb{G}_{AB} \partial_\tau^2 h^B \ , \quad \mathbb{G}^{ij,kl} = \frac{1}{8}(g^{ik}g^{jl} + g^{il}g^{jk}) - \frac{\lambda}{4} g^{ij} g^{kl} \ , \tag{18}$$

where $h^A \equiv h_{ij}$. The part of the quadratic action with space derivatives of the metric is too lengthy to be written explicitly. Schematically, it has the form,

$$\mathcal{L}_{\text{pot, hh}} + \mathcal{L}_{\text{gf},\, hh} = \frac{\sqrt{g}}{2G} \underbrace{h^A \mathbb{D}_{AB} h^B}_{\approx\, 400 \text{ terms}}, \tag{19}$$

where $\mathbb{D}_{AB}$ is a purely 3-dimensional differential operator of 6th order. In flat background it reduces to terms with exactly 6 derivatives,

$$h^A \mathbb{D}_{AB} h^B = \left(\frac{\nu_5}{2} - \frac{1}{4\sigma}\right) h^{ik} \Delta^2 \partial_i \partial_j h^{jk} + \left(2\nu_4 + \frac{\nu_5}{2} + \frac{\lambda(1+\xi)}{2\sigma}\right) h \Delta^2 \partial_k \partial_l h^{kl}$$
$$- \left(\nu_4 + \frac{\nu_5}{2} + \frac{\xi}{4\sigma}\right) h^{ij} \Delta \partial_i \partial_j \partial_k \partial_l h^{kl} + \left(-\nu_4 - \frac{\nu_5}{4} - \frac{\lambda^2(1+\xi)}{4\sigma}\right) h \Delta^3 h - \frac{\nu_5}{4} h^{ij} \Delta^3 h_{ij} \ .$$

One-loop action and 3D reduction

Effective action consists of two parts

$$\Gamma^{1-\text{loop}} = \frac{1}{2}\mathrm{Tr}\ln(-\delta^A_B\partial^2_\tau + \mathbb{D}^A{}_B) - \frac{1}{2}\mathrm{Tr}\ln\left(-\delta^i_j\partial^2_\tau + \mathbb{B}^i{}_j\right) \, , \tag{20}$$

where

$$\mathbb{D}^A{}_B = (\mathbb{G}^{-1})^{AC}\mathbb{D}_{CB}. \tag{21}$$

The full one-loop action can be expressed as

$$\Gamma^{1-\text{loop}} = \frac{1}{2}\int d\tau \left[\mathrm{Tr}_3\sqrt{\mathbb{D}^A{}_B} - \mathrm{Tr}_3\sqrt{\mathbb{B}^i{}_j}\right]. \tag{22}$$

The strategy for evaluation

The operators $\mathbb{F} = (\mathbb{D}, \mathbb{B})$ can be brought into the form:

$$\mathbb{F} = \sum_{a=0}^{6} \mathcal{R}_{(a)} \sum_{6 \geq 2k \geq a} \alpha_{a,k} \nabla_1 \dots \nabla_{2k-a} (-\Delta)^{3-k}\,, \quad \mathcal{R}_{(a)} = O\Big(\frac{1}{l^a}\Big). \tag{23}$$

Their square roots are nonlocal pseudo-differential operators given by

$$\sqrt{\mathbb{F}} = \sum_{a=0}^{\infty} \mathcal{R}_{(a)} \sum_{k \geq a/2}^{K_a} \tilde{\alpha}_{a,k} \nabla_1 \dots \nabla_{2k-a} \frac{1}{(-\Delta)^{k-3/2}}\,, \tag{24}$$

The UV divergent part of $\varGamma^{1-\mathrm{loop}}$ follows from the calculation of UFTs

$$\int d^3x\, \mathcal{R}_{(a)}(\mathbf{x}) \nabla_1 \dots \nabla_{2k-a} \frac{1}{(-\Delta)^{k-3/2}} \delta(\mathbf{x}, \mathbf{x}') \Big|_{\mathbf{x}=\mathbf{x}'}. \tag{25}$$

Since the divergences of HG have at maximum the dimensionality $a = 6$, only finite number of such traces will be needed. The problem is split in two steps — calculation of the operator square root and the evaluation of UFTs.

Perturbative scheme

Square root contains all powers of curvature. By denoting all curvature corrections in $\sqrt{\mathbb{F}}$ as $\mathbb{X}$,

$$\sqrt{\mathbb{F}} = \mathbb{Q}^{(0)} + \mathbb{X} \tag{26}$$

one obtains the equation for this correction term

$$\mathbb{Q}^{(0)}\mathbb{X} + \mathbb{X}\,\mathbb{Q}^{(0)} = \mathbb{F} - \left(\mathbb{Q}^{(0)}\right)^2 - \mathbb{X}^2, \quad \mathbb{F} - \left(\mathbb{Q}^{(0)}\right)^2 \propto R. \tag{27}$$

This nonlinear equation can be solved by iterations because its right hand side is at least linear in curvature.

Universal functional traces

I.Jack and H.Osborn (1984), A.Barvinsky and G.Vilkovisky (1985)

$$\nabla_{i_1}\ldots\nabla_{i_p}\frac{\hat{1}}{(-\Delta)^{N+1/2}}\delta(x,y)\Big|_{y=x}=\frac{1}{\Gamma(N+1/2)}\nabla_{i_1}\ldots\nabla_{i_p}\int_0^\infty ds\, s^{N-1/2}\,\mathrm{e}^{s\Delta}\,\hat{\delta}(x,y)\Big|_{y=x}^{\mathrm{div}}.$$

Heat-kernel (Schwinger-DeWitt) expansion

$$\mathrm{e}^{s\Delta}\,\hat{\delta}(x,y)=\frac{\mathcal{D}^{1/2}(x,y)}{(8\pi)^{d/2}}e^{-\frac{\sigma(x,y)}{2s}}\sum_{n=0}^{\infty}s^n\hat{a}_n(x,y).$$

Example of tensor UFTs

$$g^{ij}(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|_{y=x}^{\mathrm{div}}=-\frac{\ln L^2}{16\pi^2}\sqrt{g}\,g^{kl}\frac{1}{30}\left(\frac{1}{2}R^{mn}R_{mn}+\frac{1}{4}R^2+\Delta R\right),$$

$$\int d^3x\,\delta^j_l\nabla_k\nabla^i(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)\Big|_{y=x}^{\mathrm{div}}$$

$$=-\frac{\ln L^2}{16\pi^2}\int d^3x\sqrt{g}\left(-\frac{23}{80}R^i_jR^j_kR^k_i+\frac{753}{1120}R_{ij}R^{ij}R-\frac{22}{105}R^3-\frac{1}{84}R\Delta R-\frac{61}{560}R_{ij}\Delta R^{ij}\right).$$

Beta functions

Essential couplings

$$\mathcal{G} = \frac{G}{\sqrt{\nu_5}}, \qquad \lambda, \qquad u_s = \sqrt{\frac{(1-\lambda)(8\nu_4+3\nu_5)}{(1-3\lambda)\nu_5}}, \qquad v_a = \frac{\nu_a}{\nu_5}, \quad a = 1,2,3, \quad (28)$$

$$\beta_\lambda = \mathcal{G}\,\frac{27(1-\lambda)^2 + 3u_s(11-3\lambda)(1-\lambda) - 2u_s^2(1-3\lambda)^2}{120\pi^2(1-\lambda)(1+u_s)u_s}, \quad (29a)$$

$$\beta_{\mathcal{G}} = \frac{\mathcal{G}^2}{26880\pi^2(1-\lambda)^2(1-3\lambda)^2(1+u_s)^3u_s^3}\sum_{n=0}^{7} u_s^n\,\mathcal{P}_n^{\mathcal{G}}[\lambda, v_1, v_2, v_3], \quad (29b)$$

$$\beta_\chi = A_\chi\,\frac{\mathcal{G}}{26880\pi^2(1-\lambda)^3(1-3\lambda)^3(1+u_s)^3u_s^5}\sum_{n=0}^{9} u_s^n\,\mathcal{P}_n^{\chi}[\lambda, v_1, v_2, v_3], \quad (29c)$$

where the prefactor coefficients $A_\chi = (A_{u_s}, A_{v_1}, A_{v_2}, A_{v_3})$ equal

$$A_{u_s} = u_s(1-\lambda), \quad A_{v_1} = 1, \quad A_{v_2} = A_{v_3} = 2. \quad (30)$$

Example of a polynomial

$$\mathcal{P}_2^{u_s} = -2(1-\lambda)^3\big[2419200v_1^2(1-\lambda)^2 + 8v_2^2(42645\lambda^2 - 86482\lambda + 43837) + v_3^2(58698 - 106947\lambda + 48249\lambda^2) + 4032v_1\big(462v_2(1-\lambda)^2 + 201v_3(1-\lambda)^2 + 30\lambda^2 - 44\lambda - 10\big) + 8v_2(6252\lambda^2 - 9188\lambda - 1468) + 8v_2v_3(34335\lambda^2 - 71196\lambda + 36861) + v_3(20556\lambda^2 - 30792\lambda - 3696) + 4533\lambda^2 - 3881\lambda + 1448\big].$$

Fixed points of RG flow

There are 5 solutions for the system of equations

$$\beta_{g_i} = 0 \,, \qquad g_i = \lambda, u_s, v_1, v_2, v_3 \,. \tag{31}$$

They written down in the table

λ	u_s	v_1	v_2	v_3	$\beta_{\mathcal{G}}/\mathcal{G}^2$	AF?
0.1787	60.57	-928.4	-6.206	-1.711	-0.1416	yes
0.2773	390.6	-19.88	-12.45	2.341	-0.2180	yes
0.3288	54533	3.798×10^8	-48.66	4.736	-0.8484	yes
0.3289	57317	-4.125×10^8	-49.17	4.734	-0.8784	yes
0.333332	3.528×10^{11}	-6.595×10^{23}	-1.950×10^8	4.667	-3.989×10^6	yes

Invariance of GR under 4d diffeomorphisms sets the value of λ to 1. That's why one expects that $\lambda \to 1^+$ in the IR limit. However, all the solutions lie on the left side of the unitary domain

$$\lambda < 1/3 \quad \text{or} \quad \lambda > 1 \tag{32}$$

and there are no RG trajectories with $\lambda \to 1^+$.

$\lambda \to \infty$ limit

A.Gümrükçüoğlu and S.Mukohyama, Rev. D 83 (2011) 124033

The beta function β_λ diverges in the limit $\lambda \to \infty$. For the new variable ϱ, the limit $\lambda = \infty$ corresponds to the finite $\varrho = 1$. It's beta function reads

$$\beta_\varrho = 3(1-\varrho)\mathcal{G}\frac{2u_s^2 + u_s\varrho(4-5\varrho) - 3\varrho^2}{40\pi^2 u_s(1+u_s)\varrho}, \quad \varrho \equiv 3\frac{1-\lambda}{1-3\lambda}. \tag{33}$$

Solutions of the system

$$\beta_\chi/\mathcal{G}\Big|_{\substack{\lambda=\infty \\ (\varrho=1)}} = 0\,, \qquad \chi = u_s, v_1, v_2, v_3\,. \tag{34}$$

are written down in the table

№	u_s	v_1	v_2	v_3	$\beta_\mathcal{G}/\mathcal{G}^2$	AF?	Can flow out of $\varrho = 1$?
1	0.0195	0.4994	-2.498	2.999	-0.2004	yes	no
2	0.0418	-0.01237	-0.4204	1.321	-1.144	yes	no
3	0.0553	-0.2266	0.4136	0.7177	-1.079	yes	no
4	12.28	-215.1	-6.007	-2.210	-0.1267	yes	yes
5 (A)	21.60	-17.22	-11.43	1.855	-0.1936	yes	yes
6 (B)	440.4	-13566	-2.467	2.967	0.05822	no	yes
7	571.9	-9.401	13.50	-18.25	-0.0745	yes	yes
8	950.6	-61.35	11.86	3.064	0.4237	no	yes

Stability matrix

In the vicinity of a fixed point, the linearized RG flow can be analyzed with the help of the stability matrix ${B_i}^j$,

$$\tilde{\beta}_{g_i} \cong \sum_j {B_i}^j (g_j - g_j^*), \quad {B_i}^j \equiv \left(\frac{\partial \tilde{\beta}_{g_i}}{\partial g_j} \right) \Big|_{g_i = g_i^*}, \quad \tilde{\beta}_{g_i} = \beta_{g_i} / \mathcal{G}, \tag{35}$$

where g_i^* are fixed point values of the coupling constants.

Fixed point	θ^1	θ^2	θ^3	θ^4	θ^5
A	-0.0141	-0.0700	0.257	0.320	0.0657
B	-0.0151	0.603	0.308	$0.092 \pm 0.289\, i$	

Table: Eigenvalues θ^I of the stability matrix for the fixed points A and B.

RG equation

We choose as an initial condition of the RG equation a point slightly shifted from the fixed point g^* in the repulsive direction

$$\begin{cases} \dfrac{dg_i}{d\tau} = \tilde{\beta}_{g_i}, \quad g_i = (v_1, v_2, v_3, u_s, \varrho), \\ g_i(0) = g_i^* + \varepsilon\, c_J\, w_i^J, \quad J = 1, 2, 3, 4, 5. \end{cases} \tag{36}$$

where ε is a small parameter, c_J are constants and w_i^J are eigenvectors enumerated by the index J, $B_i^{\ j} w_j^J = \theta^J w_i^J$, with $\theta^J < 0$.

Eigen-vector	w_ϱ	w_{v_1}	w_{v_2}	w_{v_3}	w_{u_s}
$A1$	0.0423	-0.0398	5.25×10^{-3}	5.57×10^{-3}	0.998
$A2$	0	-0.115	-0.224	0.0480	-0.967
$B1$	2.19×10^{-5}	-0.999	1.87×10^{-5}	5.69×10^{-6}	0.0162

Table: Eigenvectors of the stability matrix with negative eigenvalues for the fixed points A and B.

From A to B

First we build the trajectory flowing from point A along the eigenvector $A2$. Since this vector has zero ϱ-part, the trajectory stays in the hyperplane $\varrho = 1$.

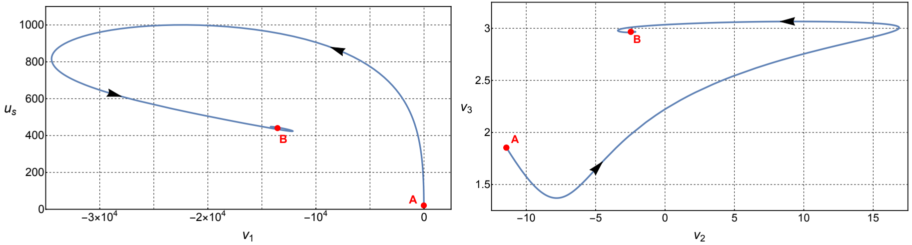

Figure: RG trajectory connecting fixed points A and B. The trajectory lies entirely in the hyperplane $\varrho = 1$. Panels show its projections on the (u_s, v_1) and (v_2, v_3) planes. Arrows indicate the flow from UV to IR.

From B to $\lambda \to 1^+$

Point B has a unique repulsive direction, pointing away from the $\varrho = 1$ hyperplane. This gives rise to two RG trajectories, depending on the sign of ε in the initial conditions.

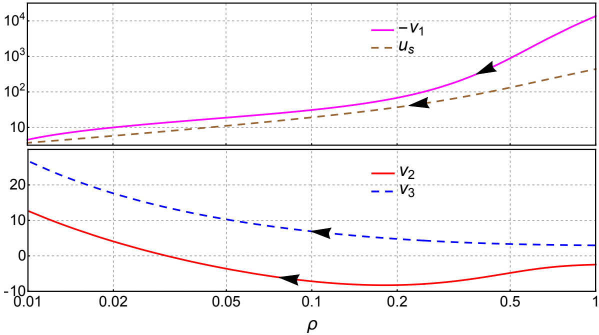

Figure: The couplings (u_s, v_a) as functions of ϱ along the RG trajectory from the fixed point B to $\varrho = 0$ ($\lambda \to 1^+$). Arrows indicate the flow from UV to IR.

The behaviour of $\mathcal{G}$

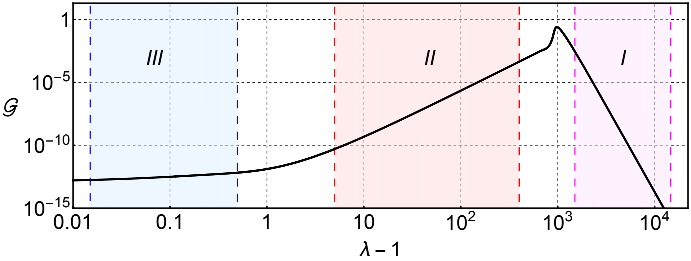

Figure: Behaviour of $\mathcal{G}$ as a function of $(\lambda - 1)$ along an RG trajectory connecting the point A to $\lambda \to 1^+$. In regions I, II and III the dependence is well described by the power law $\mathcal{G} \propto (\lambda - 1)^k$ with $k_I = -13.69$, $k_{II} = 3.84$, $k_{III} \approx 0.37$.

Conclusions

- Beta functions for all essential coupling in (3+1)-dimensional Hořava gravity were obtained.
- The results underwent a number of very powerful checks.
- All the fixed points of RG flow were found.
- A family of trajectories was found. They are connecting AF fixed point in the UV to the region where the kinetic term has GR form.