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dedicated to Prof. Rubakov memory
Alikhanian National Laboratory (Yerevan Physics Institute)

On the strong coupling problem in cosmologies with "strong gravity in the past"

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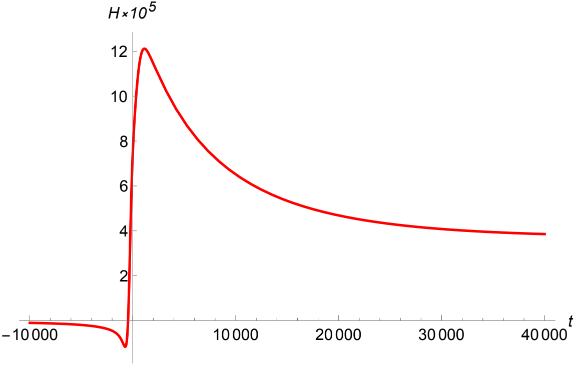
Motivation

We study different alternative/completion scenarios to inflation (A.Starobinsky; A.Guth; A.Linde; K.Sato), because:

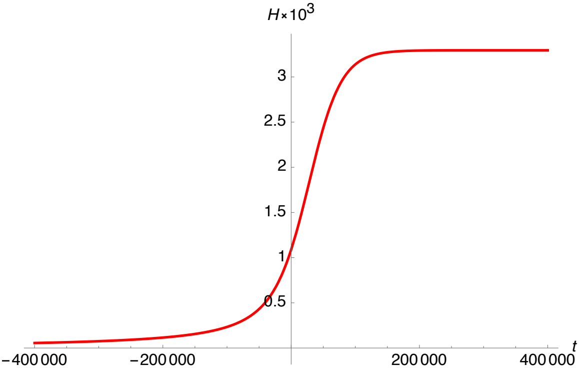
- inflation has some long-lasting conceptual problems, namely $\rightarrow$ geodesic incompleteness (A.Borde, A.Vilenkin);
- if we rule alternative scenarios out $\rightarrow$ it can be another indirect confirmation of inflation paradigm.

The Universe with bounce



T.Qui’2011,2013; D.Easson’2011; Y.Cai’2012;
M.Osipov’2013; M.Koehn’2013; L.Battarra’2014; A.Ijjas’2016;
YA, P.Petrov, V.Rubakov’2021

Genesis



P.Creminelli, A.Nicolis, E.Trincherini'2010;
YA, P.Petrov, V.Rubakov'2021

Non-singular evolution
Null Energy Condition

Realization of non-singular evolution within classical field theory requires the violation of the Null Energy Condition (NEC) $T_{\mu\nu}n^{\mu}n^{\nu} > 0$. Consider FLRW Universe, homogeneous and isotropic matter

$$T_{00} = \rho, \quad T_{ij} = a^2\gamma_{ij}p,$$

$$\dot{H} = -4\pi G(\rho + p) + \sout{\text{curvature term}}.$$

$$\frac{d\rho}{dt} = -3H(\rho + p).$$

Use $n_{\mu} = (1, a^{-1}\nu^{i})$ with $\gamma_{ij}\nu^{i}\nu^{j} = 1$ and then NEC leads to

$$T_{\mu\nu}n^{\mu}n^{\nu} > 0 \rightarrow \rho + p > 0.$$

Penrose theorem: singularity in the past.

Horndeski theory

G. W. Horndeski'1974

Full Horndeski Lagrangian:

$$\begin{aligned}\mathcal{L} = {} & \mathrm{G}_2(\phi, \mathrm{X}) - \mathrm{G}_3(\phi, \mathrm{X})\Box\phi + \\ & \mathrm{G}_4(\phi, \mathrm{X})\mathrm{R} + \mathrm{G}_{4,\mathrm{X}}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ & + \mathrm{G}_5(\phi, \mathrm{X})\mathrm{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\ & - \frac{1}{6}\mathrm{G}_{5,\mathrm{X}}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right],\end{aligned}$$

where,

$$\mathrm{X} = -\frac{1}{2}\mathrm{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

$$\Box\phi = \mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu\phi.$$

For our purposes, it is enough to study a subclass of this theory:

$$\mathcal{L} = \mathrm{G}_2(\phi, \mathrm{X}) - \mathrm{G}_3(\phi, \mathrm{X})\Box\phi + \mathrm{G}_4(\phi)\mathrm{R}.$$

Stability of the model
No-Go theorem

Let us consider the following perturbed ADM metric:

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right),$$

with

$$\gamma_{ij} = a^2(t) e^{2\zeta}(\delta_{ij} + h_{ij} + \ldots), \quad N = N_0(t)(1+\alpha), \quad N_i = \partial_i \beta.$$

where $h_{ii} = 0, \partial_i h_{ij} = 0$. Here α and β are not physical. We work with unitary gauge $\delta\phi = 0$. The quadratic actions for ζ and h_{ij} are given, respectively:

$$\mathcal{S}_{hh} = \int dt d^3x \frac{Na^3}{8}\left[\mathcal{G}_T \frac{\dot{h}_{ij}^2}{N^2} - \frac{\mathcal{F}_T}{a^2} h_{ij,k} h_{ij,k}\right],$$

$$\mathcal{S}_{ss} = \int dt d^3x Na^3 \left[\mathcal{G}_S \frac{\dot{\zeta}^2}{N^2} - \frac{\mathcal{F}_S}{a^2} \zeta_{,i}\zeta_{,i}\right].$$

Stability of the model

No-Go theorem

- We consider $a(t) > \text{const} > 0$;
- No-Go statement: $\mathcal{F}_{S,T} < 0$ at some moment of time, instability in bounce/genesis models, M. Libanov, S. Mironov, V. Rubakov'2016; T. Kobayashi'2016; S. Mironov, V. Rubakov, V. Volkova'2018;
- Avoid No-Go:

$$\int_{-\infty}^{t} a(t)(\mathcal{F}_T + \mathcal{F}_S)dt < \infty \, .$$

Strong coupling problem

Validity of classical description?

- One way is to go beyond Horndeski and DHOST [Y.Cai et.al.' 2016, P.Creminelli et.al.'2016, R.Kolevatov et.al.'2017, Y. Cai, Y. Piao'2017; S.Mironov, V.Rubakov, V.Volkova'19,'20,'22]
- Another way to avoid No-Go theorem for Horndeski is to obtain such a model/solution that $\mathcal{F}_{\mathrm{S,T}}$ coefficients have asymptotics

$$\mathcal{F}_{\mathrm{S,T}} \to 0 \text{ as } t \to -\infty, \text{ where } \mathcal{F}_{\mathrm{T}} = 2\mathrm{G}_4.$$

- This means that

$$\mathrm{G}_4 \to 0 \text{ as } t \to -\infty.$$

- The effective Planck mass (recall that $\Delta\mathcal{L} = \mathrm{G}_4(\phi)\mathrm{R}$ and also $\mathrm{G_N} = \frac{1}{\mathrm{M_{Pl}^2}}$) goes to zero, and it signalizes that we may have strong coupling at $t \to -\infty$.

Solution: no SC regime at $t \to -\infty$ in some region of lagrangian parameters.

The solution of strong coupling problem
Dimensional analysis

- The criteria of validity of classical description: Classical energy scales $\ll$ Strong coupling energy scales.
- Characteristic classical energy scale $\rightarrow$ background: $E_{class} \sim H$, (or $\frac{\dot{H}}{H}$, or $\frac{\dot{\phi}}{\phi}$).
- Strong coupling energy scales $\rightarrow$ dimensional analysis of couplings in higher order action (interactions among scalar and tensor perturbations):

$$\mathbb{L}^{(i)}_{\zeta\zeta\zeta} \sim \hat{\Lambda}_i \cdot \pi^3 \cdot (\partial_t)^{a_i} \cdot (\partial)^{b_i}, \quad [\hat{\Lambda}_i] = 1 - a - b,$$

$$E^{\zeta\zeta\zeta,(i)}_{strong} \sim \hat{\Lambda}_i^{-\frac{1}{a_i+b_i-1}},$$

- It was shown, that the classical treatment of the background is legitimate at $t \rightarrow -\infty$ in a certain range of parameters.

YA, O.Evseev, O.Melichev, V.Rubakov'2018,2020;
YA, P.Petrov, V.Rubakov'2020,2021

On the strong coupling problem in cosmologies with "strong gravity in the past"

- We examine the potential strong coupling problem at early times in a contracting cosmological model with "strong gravity in the past" (Jordan frame), which is conformally related to inflation (Einstein frame).
- From naive dimensional analysis in the Jordan frame one would conclude that the quantum strong coupling energy scale can be lower than the classical energy scale.
- However, from the Einstein frame prospective, this should not be the case.
- Calculation in the Jordan frame shows cancellations of the dangerous contributions in the tree level amplitude!

Contraction conformally related to inflation

- We consider a class of contracting models (Jordan frame) that are conformally related to cosmological inflation.
- The action in the Jordan (contraction) frame is given by

$$\mathcal{S}_b = \int d^4x\sqrt{-g}\left[P(\phi, X) + \frac{M_P^2 f^2(\phi)}{2}R\right] ,$$

with

$$P(\phi, X) = \omega(\phi)X - V(\phi),$$

$$\omega(\phi) = f^2 - 6M_P^2\left(\frac{df}{d\phi}\right)^2 , \qquad V(\phi) = f^4(\phi)V_I(\phi) .$$

- By conformal transformation

$$g_{\mu\nu} = f^{-2}(\phi)g_{I\,\mu\nu}$$

the theory with S_b is related to the following inflationary model in the Einstein (inflation) frame:

$$S_I = \frac{1}{2}\int d^4x\sqrt{-g_I}\left[M_P^2 R_I - g_I^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V_I(\phi)\right] .$$

Inflation in Einstein frame

- We consider inflation potential that flattens out at large fields,

$$V_I(\phi) \to V_\infty, \quad \text{as} \quad \phi \to \infty\,; \qquad V_\infty \ll M_P^4\,,$$

- Inflation occurs in the slow roll regime at early times ($\epsilon, \eta \ll 1$):

$$\epsilon = \frac{(V_I')^2 M_P^2}{2V^2}\,, \quad \eta = \frac{V_I'' M_P^2}{V}\,.$$

- The slow roll equations are

$$\frac{d\phi(\tau)}{d\tau} = -\frac{M_P V_I'}{\sqrt{3V_I}}\,, \qquad H_I = \sqrt{\frac{V_I}{3}}\frac{1}{M_P}\,,$$

  where τ is a cosmic time in this frame.
- For asymptotically flat inflaton potentials, one typically has $\eta \gg \epsilon$ (or $V_I V_I'' \gg (V_I')^2$).

Contraction in Jordan frame

- Choose the function defining the conformal transformation as follows

$$f(\phi) = f_0 \exp\left[-\frac{(\alpha+1)}{M_P^2}\int d\phi \frac{V_I}{V_I'}\right], \quad \alpha > 0.$$

- Then the Jordan frame metric is

$$ds^2 = f^{-2}(\phi(\tau))d\tau^2 - f^{-2}(\phi(\tau))a_I^2(\tau)dx^2,$$

where $a_J \equiv f^{-1}a_I$ is corresponding scale factor in Jordan frame, and the differential of cosmic time in this frame is given by $f^{-1}d\tau$.

- So, the Hubble parameter in the Jordan frame is given by

$$H = f\frac{d}{d\tau}\ln(a_I f^{-1}) = -f\cdot\frac{\alpha}{M_P}\sqrt{\frac{V_I}{3}}\ .$$

The terms in the cubic action for scalars, which do not vanish in our model are:

$$\mathcal{S}^{(3)} = \int \mathrm{dt d^3x\, a^3} \left\{ \mathcal{C}_1 \zeta \dot{\zeta}^2 + \frac{1}{\mathrm{a}^2} \mathcal{C}_2 \zeta (\partial\zeta)^2 + \mathcal{C}_4 \dot{\zeta} \left(\partial_{\mathrm{i}}\zeta\right) \left(\partial_{\mathrm{i}}\mathcal{X}\right) + \mathcal{C}_5 \partial^2\zeta (\partial\mathcal{X})^2 \right\} ,$$

where $\partial^2 = \partial_{\mathrm{i}}\partial_{\mathrm{i}}$ and $\partial^2\mathcal{X} = \dot{\zeta}$

$$\mathcal{C}_1 = \mathrm{f}^2 \cdot \frac{\mathrm{M_P^6 (V_I')^2}}{\mathrm{4V_I^4}} \left(\mathrm{4V_I V_I''} - 3(\mathrm{V_I'})^2\right),$$

$$\mathcal{C}_2 = \mathrm{f}^2 \cdot \frac{\mathrm{M_P^6 (V_I')^2}}{\mathrm{4V_I^4}} \left(5(\mathrm{V_I'})^2 - \mathrm{4V_I V_I''}\right),$$

$$\mathcal{C}_4 = \mathrm{f}^2 \frac{\mathrm{M_P^6 (V_I')^4}}{\mathrm{16V_I^6}} \left(\mathrm{M_P^2 (V_I')^2 - 8V_I^2}\right),$$

$$\mathcal{C}_5 = \mathrm{f}^2 \frac{\mathrm{M_P^8 (V_I')^6}}{\mathrm{32V_I^6}} .$$

- The classical energy scale is of order of the Hubble parameter:

$$|E^{(\mathrm{class})}| = |H| \sim \frac{f\sqrt{V_I}}{M_P} \ .$$

- To obtain an estimate of the strong coupling scale through naive dimensional analysis, we set, at a given moment of time, $a = 1$ and introduce canonically normalized field $\zeta_c = \sqrt{2\mathcal{G}_S}\zeta$.
- In terms of the canonically normalized field, the cubic action still has the previous form with the replacement

$$\tilde{\mathcal{C}}_i = (2\mathcal{G}_S)^{-3/2}\mathcal{C}_i \ ,$$

so that

$$\tilde{\mathcal{C}}_1 = \frac{1}{f} \cdot \frac{(-3(V_I')^2 + 4V_I V_I'')}{4V_I V_I'}, \quad \tilde{\mathcal{C}}_2 = \frac{1}{f} \cdot \frac{(5(V_I')^2 - 4V_I V_I'')}{4V_I V_I'},$$

$$\tilde{\mathcal{C}}_4 \sim \frac{1}{f} \cdot \frac{V_I'}{V_I} \ , \qquad \tilde{\mathcal{C}}_5 \sim \frac{1}{f} \cdot M_{Pl}^2 \left(\frac{V_I'}{V_I}\right)^3 \ .$$

Strong coupling
Dimensional analysis

- All operators in the resulting cubic Lagrangian are dimension-5, so one immediately finds naive estimates for the associated strong coupling scales
$$\mathrm{E}_{\mathrm{i}}^{(\mathrm{naive})} \sim |\tilde{\mathcal{C}}_{\mathrm{i}}|^{-1} \,.$$
- Naively, the most relevant of these scales are the lowest ones, which are associated with the largest $\mathcal{C}_{\mathrm{i}}$.
- The two naive strong coupling scales (coming from $\mathcal{C}_{1,2}$) are of the same order:
$$\mathrm{E}^{(\mathrm{naive})} \sim \mathrm{f}\frac{\mathrm{V}_{\mathrm{I}}'}{\mathrm{V}_{\mathrm{I}}''}.$$
- Depending on the shape of the inflaton potential, classical energy scale may exceed strong coupling energy scale! For instance, consider $\mathrm{V}_{\mathrm{I}} = \mathrm{V}_{\infty}\left(1 - \mathrm{e}^{\phi^2/\mu^2}\right)$. Arrive to
$$\frac{\mathrm{E}^{(\mathrm{naive})}}{\mathrm{E}^{(\mathrm{class})}} \sim \frac{\mu^2}{\phi \mathrm{H}_{\mathrm{I}}},$$
which is less than 1 at large ϕ. Strong coupling regime?

- If not for the Einstein frame considerations, one would be tempted to dismiss such a model!
- However, using amplitudes analysis one finds that there are strong cancellations. Indeed, it is straightforward to calculate $2 \to 2$ scattering amplitude.
- But firstly, we note, if we set, e.g. $\tilde{\mathcal{C}}_2 = 0$, then the matrix element would be given by

$$\mathrm{M}_{\tilde{\mathcal{C}}_1\,;\,\tilde{\mathcal{C}}_2=0} = -\frac{\mathrm{E}^2}{\mathrm{f}^2} \cdot \frac{(9\mathrm{x}^2-5)\big(3(\mathrm{V}_\mathrm{I}')^2 - 4\mathrm{V}_\mathrm{I}\mathrm{V}_\mathrm{I}''\big)^2}{64(\mathrm{x}^2-1)\mathrm{V}_\mathrm{I}^2(\mathrm{V}_\mathrm{I}')^2}\,, \mathrm{x} \equiv \cos\theta,$$

so the partial wave amplitudes

$$\mathrm{a}^{(\mathrm{l})} = \frac{1}{32\pi}\int \mathrm{dx}\ \mathrm{P}_\mathrm{l}(\mathrm{x})\,\mathrm{M}_{\tilde{\mathcal{C}}_1\,;\,\tilde{\mathcal{C}}_2=0},$$

would hit the unitarity bound $|\mathrm{a}^{(\mathrm{l})}| = 1/2$ (C.Grojean'07) at $\mathrm{E} \sim \mathrm{E}^{(\mathrm{naive})}$.

Strong coupling
Analysis of the amplitudes

- However, there are strong cancellations. Indeed,

$$\mathrm{M_s} = -\frac{\mathrm{E}^2}{4}(3\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2)^2 \,, \quad \mathrm{M_t} = \frac{\mathrm{E}^2}{2(1-\mathrm{x})}\left[\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2(2-\mathrm{x})\right]^2 \,,$$

$$\mathrm{M_u} = \frac{\mathrm{E}^2}{2(1+\mathrm{x})}\left[\tilde{\mathcal{C}}_1 + \tilde{\mathcal{C}}_2(2+\mathrm{x})\right]^2 \,,$$

$$\mathrm{M} = \mathrm{M_s} + \mathrm{M_t} + \mathrm{M_u} = \frac{\mathrm{E}^2}{\mathrm{f}^2} \cdot \frac{(41\mathrm{x}^2 - 45)(\mathrm{V}_\mathrm{I}')^2 - 40(\mathrm{x}^2-1)\mathrm{V_I}\mathrm{V}_\mathrm{I}''}{16(\mathrm{x}^2-1)\mathrm{V}_\mathrm{I}^2} \,.$$

- We see that the strong coupling scale is actually given by (recalling $\mathrm{V_I}\mathrm{V}_\mathrm{I}'' \gg (\mathrm{V}_\mathrm{I}')^2$)

$$\mathrm{E}^{(\mathrm{strong})} \sim \mathrm{f} \cdot \left(\frac{\mathrm{V_I}}{\mathrm{V}_\mathrm{I}''}\right)^{1/2} \sim \mathrm{f} \cdot \frac{\mathrm{M_{Pl}}}{\eta^{1/2}} \,.$$

- As anticipated, this scale is much higher than the classical energy scale $(\mathrm{f}\sqrt{\mathrm{V_I}})/\mathrm{M_{Pl}}$ for $\mathrm{V_I} \ll \mathrm{M_{Pl}^4}$.

Unitarity relation and unitarity bounds for scalars with different sound speeds

- Only scalars were considered - However, in cosmological models we have different types of perturbations!
- Thus, we consider a theory which contains massless scalar fields with different sound speeds.
- We derive unitarity relations for partial wave amplitudes of $2 \rightarrow 2$ scattering, with explicit formulas for contributions of two-particle intermediate states.
- Making use of these relations, we obtain unitarity bounds both in the most general case and in the case considered in literature for unit sound speed.
- These bounds can be used for estimating the strong coupling scale of a pertinent EFT.

Unitarity relation and unitarity bounds for scalars with different sound speeds

- The quadratic action reads

$$S = \sum_i S_{\phi_i}\,, \quad S_{\phi_i} = \int d^4x \left(\frac{1}{2}\dot{\phi}_i^{\,2} - \frac{1}{2}u_i^2(\vec{\nabla}\phi_i)^2 \right).$$

- Now, we consider an initial state:

$$|\psi, \beta\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}a^\dagger_{p_1}a^\dagger_{p_2}\,|0\rangle\,,$$

  and the same form we have for the final state $|\psi', \beta'\rangle$.
- Notation β refers to the types of the two particles, $\beta = \{\phi_i, \phi_j\}$, while notation ψ is a shorthand for the pair of momenta, $\psi = \{\vec{p}_1, \vec{p}_2\}$.

Unitarity relation and unitarity bounds for scalars with different sound speeds

- Unitarity of S-matrix, $SS^\dagger = S^\dagger S = 1$ implies

$$T - T^\dagger = iTT^\dagger = iT^\dagger T\,.$$

- In terms of two-particle state of definite angular momentum one has

$$-i\left(T^{(l)}_{m'\beta';m\beta} - T^{(l)\,*}_{m\beta;m'\beta'}\right) = \int d^4\mathcal{P}'' \sum_{m'',\beta''} \frac{1}{N(\beta'')} T^{(l)}_{m'\beta';m''\beta''} T^{(l)\,*}_{m\beta;m''\beta''},$$

- Then, finally, in terms of PWA

$$-\frac{i}{2}\left(a_{l,\alpha\beta} - a^*_{l,\beta\alpha}\right) = \sum_\gamma \frac{2}{u_{1\gamma}u_{2\gamma}(u_{1\gamma}+u_{2\gamma})} a_{l,\alpha\gamma}a^*_{l,\beta\gamma},$$

$$\text{Im}\, a_{l,\alpha\beta} = \sum_\gamma \frac{2}{u_{1\gamma}u_{2\gamma}(u_{1\gamma}+u_{2\gamma})} a_{l,\alpha\gamma}a^*_{l,\gamma\beta}, \quad \text{distinguishable particles,}$$

$$\text{Im}\, a_{l,\alpha\beta} = \sum_\gamma \frac{1}{2u_\gamma^3} a_{l,\alpha\gamma}a^*_{l,\gamma\beta}, \quad \text{identical particles.}$$

Unitarity relation and unitarity bounds for scalars with different sound speeds

- Finally, the generalization of the PWA unitarity relation is

$$\mathrm{Im}\ \mathrm{a}_{\mathrm{l},\alpha\beta} = \sum_{\gamma} \mathrm{a}_{\mathrm{l},\alpha\gamma} \frac{\mathrm{g}_\gamma}{\mathrm{u}_{\gamma 1}\mathrm{u}_{\gamma 2}(\mathrm{u}_{\gamma 1} + \mathrm{u}_{\gamma 2})} \mathrm{a}^*_{\mathrm{l},\gamma\beta} \ ,$$

  where $\mathrm{g}_\gamma = 2$ if these intermediate particles are distinguishable and $\mathrm{g}_\gamma = 1$ if these particles are identical.
- Upon redefining

$$\tilde{\mathrm{a}}^{(\mathrm{l})}_{\alpha\beta} = \left(\frac{\mathrm{g}_\alpha}{\mathrm{u}_{\alpha 1}\mathrm{u}_{\alpha 2}(\mathrm{u}_{\alpha 1} + \mathrm{u}_{\alpha 2})}\right)^{1/2} \mathrm{a}^{(\mathrm{l})}_{\alpha\beta} \left(\frac{\mathrm{g}_\beta}{\mathrm{u}_{\beta 1}\mathrm{u}_{\beta 2}(\mathrm{u}_{\beta 1} + \mathrm{u}_{\beta 2})}\right)^{1/2} ,$$

$$\mathrm{Im}\ \tilde{\mathrm{a}}^{(\mathrm{l})} = \tilde{\mathrm{a}}^{(\mathrm{l})}\tilde{\mathrm{a}}^{(\mathrm{l})\dagger} \ .$$

- The most stringent tree level unitarity bound is obtained for the largest eigenvalue of the tree level matrix $\tilde{\mathrm{a}}^{(\mathrm{l})}$ (which is real and symmetric). This bound reads

$$|\text{maximum eigenvalue of } \tilde{\mathrm{a}}^{(\mathrm{l})}| \leq \frac{1}{2} \ .$$

Conclusion and outlook

- It was shown that naive dimensional analysis of strong coupling provides wrong results... One had better to use the analysis of diagrams/amplitudes in order to find strong coupling energy scale.
- Useful (for cosmological context) unitarity relations and bound were obtained in the theory which contains massless scalar fields with different sound speeds.
- An application of unitarity bound for real cosmological model of early Universe without initial singularity $\rightarrow$ Pavel Petrov talk!
- Even some inflation models (e.g. some class of k-inflation) may suffer from strong coupling $\rightarrow$ accurate investigation of this problem (YA, P. Petrov, in preparation), new constraints on k-inflation model?

Thank you for your attention!

Let us consider general relativity

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

in the cosmological context with

$$ds^2 = dt^2 - a^2\gamma_{ij}dx^i dx^j,$$

and isotropic and homogeneous matter, filling the Universe

$$T_{00} = \rho, \quad T_{ij} = a^2\gamma_{ij}p,$$

then it follows from the combinations of (00) and (ij) components of the Einstein equations that

$$\dot{H} = -4\pi G(\rho + p) + \text{curvature term},$$

where H is the Hubble parameter.

$$\dot{H} = -4\pi G(\rho + p) + \text{curvature term},$$

An important characteristic here is the null energy condition (NEC) for the matter energy-momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} k^\mu k^\nu \geq 0,$$

for every null vector k^μ. Let us use $k_\mu = (1, a^{-1}\nu^i)$ with $\gamma_{ij}\nu^i\nu^j = 1$ and then NEC leads to

$$\rho + p \geq 0 \rightarrow \dot{H} \leq 0.$$

Another example comes from the conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \frac{d\rho}{dt} = -3H(\rho + p).$$

Null energy condition
Non-singular cosmology

- This implies that there is a singularity for H and ρ in the past of the expanding universe;
- Therefore, one either modifies gravity or violates the NEC to build non-singular cosmology;

Let's violate NEC!

However, violating the NEC in a healthy manner turns out to be challenging for any known matter.

- For canonical scalar field we have

$$\rho + \mathrm{p} = \dot{\phi}^2 \geq 0$$

  and NEC is automatically satisfied;
- Try something new, but always demand stable cosmology...

Types of instabilities

Let us consider for the simplicity the general form of the second order action for the scalar field perturbations $\phi = \phi_0 + \chi$:

$$\mathrm{L}_\chi^{(2)} = \frac{1}{2}\mathrm{U}\dot{\chi}^2 - \frac{1}{2}\mathrm{V}(\partial_\mathrm{i}\chi)^2 - \frac{1}{2}\mathrm{W}\chi^2,$$

with

$$\mathrm{U}\omega^2 = \mathrm{V}\vec{\mathrm{p}}^2 + \mathrm{W}, \quad \mathrm{c}_\chi^2 = \mathrm{V}/\mathrm{U},$$

where $\mathrm{U}, \mathrm{V}, \mathrm{W}$ are some combinations of unperturbed lagrangian functions. Here we have the following cases:

- Stable solution $\mathrm{U} > 0$, $\mathrm{V} > 0$, $\mathrm{W} \geq 0$ with $\rho > 0$.
  Use $\mathrm{U} > \mathrm{V} > 0$ to avoid superluminal propagation;
- Gradient instabilities $\mathrm{U} > 0$, $\mathrm{V} < 0$ or $\mathrm{U} < 0$, $\mathrm{V} > 0$;
- Ghost instabilities $\mathrm{U} < 0$, $\mathrm{V} < 0$.

Let us write all parts of third action straightway with the constraints imposed for scalar perturbations α and β:

$$\alpha = \frac{\mathcal{G}_T}{\Theta}\frac{\dot{\zeta}}{N},$$
$$\beta = \frac{1}{a\mathcal{G}_T}\left(a^3\mathcal{G}_S\psi - \frac{a\mathcal{G}_T^2}{\Theta}\zeta\right),$$

with $\psi := (1/N)\partial^{-2}\dot{\zeta}$. The quadratic actions for ζ and h_{ij} are given, respectively:

$$\mathcal{L}_{\zeta\zeta} = a^3\left[\mathcal{G}_S\frac{\dot{\zeta}^2}{N^2} - \frac{\mathcal{F}_S}{a^2}\zeta_{,i}\zeta_{,i}\right],$$

$$\mathcal{L}_{hh} = \frac{a^3}{8}\left[\mathcal{G}_T\frac{\dot{h}_{ij}^2}{N^2} - \frac{\mathcal{F}_T}{a^2}h_{ij,k}h_{ij,k}\right].$$

Stability of model
No-Go theorem

There $\mathcal{F}_\mathrm{S}, \mathcal{G}_\mathrm{S}, \mathcal{F}_\mathrm{T}, \mathcal{G}_\mathrm{T}$ are given by:

$$\mathcal{F}_\mathrm{T} = 2\mathrm{G}_4 + ..., \quad \mathcal{G}_\mathrm{T} = 2\mathrm{G}_4 + ...,$$

and

$$\mathcal{F}_\mathrm{S} = \frac{1}{\mathrm{a}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\mathrm{a}}{\Theta}\mathcal{G}_\mathrm{T}^2\right) - \mathcal{F}_\mathrm{T}, \quad \mathcal{G}_\mathrm{S} = \frac{\Sigma}{\Theta^2}\mathcal{G}_\mathrm{T}^2 + 3\mathcal{G}_\mathrm{T},$$

where Σ and Θ are some cumbersome expression of G_2, G_3, G_4 and H. Stability conditions are:

$$\mathcal{G}_\mathrm{T} \geq \mathcal{F}_\mathrm{T} > 0, \quad \mathcal{G}_\mathrm{S} \geq \mathcal{F}_\mathrm{S} > 0.$$

Denote $\xi = \mathrm{a}\mathcal{G}_\mathrm{T}^2/\Theta$, we rewrite $\mathcal{F}_\mathrm{S}$ as

$$\mathcal{F}_\mathrm{S} = \frac{1}{\mathrm{a}}\frac{\mathrm{d}\xi}{\mathrm{dt}} - \mathcal{F}_\mathrm{T} \rightarrow \frac{\mathrm{d}\xi}{\mathrm{dt}} > \mathrm{a}\mathcal{F}_\mathrm{T} > 0$$

Stability of model
No-Go theorem

$$\frac{d\xi}{dt} > a\mathcal{F}_T > 0, \quad \xi = a\mathcal{G}_T^2/\Theta,$$

Here $|\Theta| < \infty$ everywhere and it is smooth function of time (as it is function of ϕ and H), so ξ can never vanish (except $a = 0$) $\rightarrow$ thus we demand non-singular model. Integrating from some t_i to t_f, we obtain:

$$\xi(t_f) - \xi(t_i) > \int_{t_i}^{t_f} a(t)\mathcal{F}_T dt,$$

where $a >$ const > 0 for $t \rightarrow -\infty$ and it is increasing with $t \rightarrow +\infty$.

Stability of model

No-Go theorem

$$\xi(t_f) - \xi(t_i) > \int_{t_i}^{t_f} a(t)\mathcal{F}_T dt,$$

- Let $\xi_i < 0$, so

$$-\xi_f < |\xi_i| - \int_{t_i}^{t_f} a\mathcal{F}_T dt,$$

  where RHS $\to$ negative with $t_f \to +\infty$. So therefore $\xi_f > 0$. And it means that $\xi = 0$ at some moment of time - singularity! So we should demand $\xi > 0$ for all times.

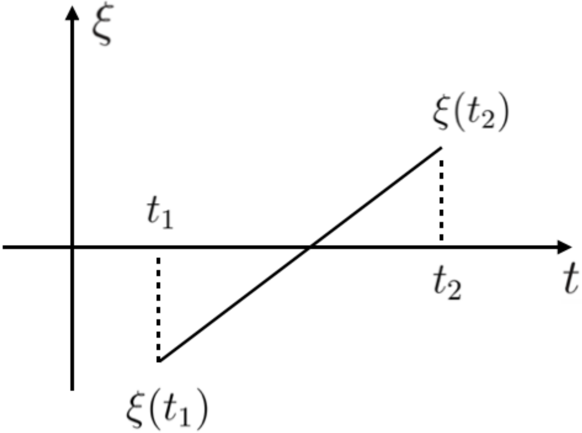
- But on the other had, again just rewritting:

$$-\xi_i > -\xi_f + \int_{t_i}^{t_f} a\mathcal{F}_T dt,$$

  and now RHS $\to$ positive with $t_i \to -\infty$ and ξ_i must be negative. Again contradiction...

Thus we have two important features here:

$$1.\xi \neq 0,$$
$$2.\mathrm{d}\xi/\mathrm{dt} > \mathrm{a}\mathcal{F}_\mathrm{T} > 0.$$



$$
\begin{aligned}
G_2 =& A_2 - 2XF_\phi, \\
G_3 =& - 2XF_X - F, \\
G_4 =& B_4,
\end{aligned}
$$

where $F(\phi, X)$ is an auxiliary function, such that

$$F_X = -\frac{A_3}{(2X)^{3/2}} - \frac{B_{4\phi}}{X},$$

Gauge is fixed by $\phi = \phi(t)$ and it gives:

$$N^{-1} = \frac{\sqrt{2X}}{\dot{\phi}(t)} \ .$$

Nonsingular cosmological models

We construct various models, namely:

- bouncing Universe which proceeds through inflationary epoch to kination (expansion within general relativity, driven by massless scalar field);
- bouncing Universe with kination stage immediately after bounce;
- combination of genesis and bounce, with the Universe starting from flat space-time, then contracting and bouncing to the expansion epoch;
- "standard" genesis evading the strong coupling problem in the past.

Bouncing Universes

In covariant formalism:

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R,$$

In ADM formalism:

$$\mathcal{L} = A_2(t, N) + A_3(t, N)K + A_4(t)(K^2 - K_{ij}^2) + B_4(t)R^{(3)}.$$

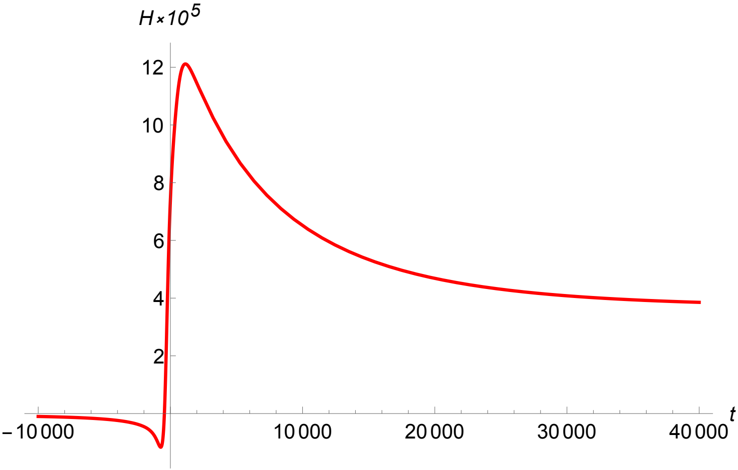
- Ansatz:

$$A_2 = \frac{1}{2}f^{-2\mu-2} \cdot a_2(t, N),$$
$$A_3 = \frac{1}{2}f^{-2\mu-1} \cdot a_3(t, N),$$
$$A_4 = -B_4 = -\frac{1}{2}f^{-2\mu},$$

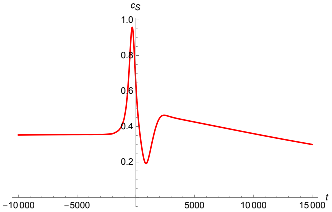
$$a_2(t, N) = \left(\frac{x(t)}{N^2} + \frac{v(t)}{N^4}\right), \quad a_3(t, N) = \frac{y(t)}{N^3}.$$

Bounce followed by inflation

Numerical example: Hubble parameter



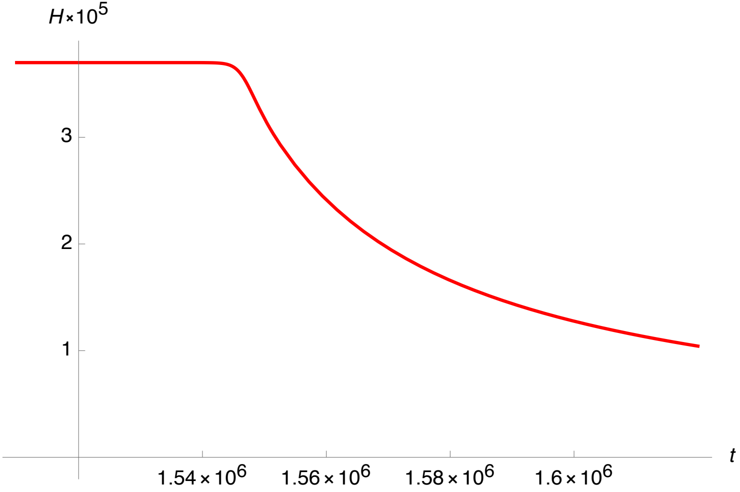
Hubble parameter at contraction, bounce, and beginning of inflation.



Contraction, bounce, and beginning of inflation.

Bounce followed by inflation

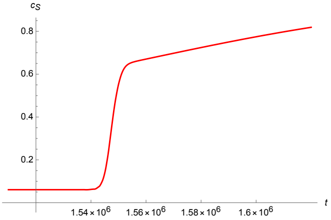
Numerical example



Hubble parameter at the end of inflation and beginning of kination.

Bounce model: from inflation to kination

Numerical example: stability for scalars



The end of inflation and beginning of kination.

Bounce followed by inflation
Numerical example

- To end up this bouncing model, we note that since the duration of inflation is fairly long, the complete expressions for all Lagrangian functions, valid at all times, are obtained by simple superpositions, i.e.

$$x(t) = x_0(1 - U_x(t)) + x_1 U_x(t)\big(1 - V(t - t_*)\big) + x_2 \frac{V(t - t_*)}{(t - t_*)^2},$$

- Depending on the parameters of the model, inflation can last for a longer or shorter time.
- Note that this property may be of interest from a phenomenological viewpoint.
- We take, quite arbitrarily, the duration of inflation approximately equal to $\Delta t_{inf} \approx 1.55 \cdot 10^6$ (in Planck units).
- It corresponds to the number of e-foldings at inflation $N_e = N_1 H_1 \Delta t_{inf} \approx 46$.

Strong coupling: dimensional analysis

- In unitary gauge, the scalar perturbation is parameterized with the field ζ, recall

$$\mathrm{ds}^2 = -[(1+\alpha)^2 - \mathrm{a}^{-2}\mathrm{e}^{-2\zeta}(\partial\psi)^2]\mathrm{dt}^2 + 2\partial_{\mathrm{i}}\psi\mathrm{dtdx}^{\mathrm{i}} + \mathrm{a}^2\mathrm{e}^{2\zeta}\mathrm{dx}^2 \;,$$

so the quadratic action for scalar perturbation is

$$\mathcal{S}^{(2)}_{\zeta\zeta} = \int \mathrm{dtd}^3\mathrm{xa}^3\mathcal{G}_{\mathrm{S}}\left[\dot{\zeta}^2 - \frac{1}{\mathrm{a}^2}\zeta_{,\mathrm{i}}\zeta_{,\mathrm{i}}\right],$$

where

$$\mathcal{G}_{\mathrm{S}} = \frac{1}{2}\frac{\dot{\phi}^2}{\mathrm{H}_{\mathrm{I}}^2} = \frac{\mathrm{f}^2}{2\mathrm{H}_{\mathrm{I}}^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = \mathrm{f}^2\cdot\frac{\mathrm{M}_{\mathrm{P}}^4(\mathrm{V}_{\mathrm{I}}')^2}{2\mathrm{V}_{\mathrm{I}}^2}\;.$$

- The perturbations propagate luminally, which is again a Jordan frame counterpart of the standard Einstein frame property.

Relations

$$\phi_i(\vec{x},t) = \int \frac{d\vec{p}_i}{(2\pi)^3}\frac{1}{\sqrt{2E_{p_i}}}\left(a_{\vec{p}_i}e^{-iE_{p_i}t+i\vec{p}_i\vec{x}} + a^\dagger_{\vec{p}_i}e^{iE_{p_i}t-i\vec{p}_i\vec{x}}\right),$$

$$E_{p_i} = u_i p_i\ , \tag{1}$$

$$[a_{\vec{p}_i'}, a^\dagger_{\vec{p}_j}] = (2\pi)^3\delta^{(3)}(\vec{p}_i' - \vec{p}_j)\delta_{ij}. \tag{2}$$

$$|\vec{p}_i\ \rangle \equiv \sqrt{2E_{p_i}}a^\dagger_{\vec{p}_i}\ |0\rangle\ ,$$

$$\langle\vec{p}\,'_j|\vec{p}\ _i\rangle = (2\pi)^3\sqrt{2E_{p_j'}2E_{p_i}}\delta^{(3)}(\vec{p}_i - \vec{p}_j')\delta_{ij}. \tag{3}$$

$$\mathbb{1} = \int \frac{d^3p_i}{(2\pi)^3 2E_{p_i}}\ |\vec{p}\ _i\rangle\ \langle\vec{p}\ _i|\ .$$

- The scalar product of states $|\psi', \beta'\rangle$ and $|\psi, \beta\rangle$ is

$$\langle \psi', \beta' \, |\psi, \beta \, \rangle = (2\pi)^6 2\mathrm{E}_{\mathrm{p}_1} 2\mathrm{E}_{\mathrm{p}_2} \delta^{(3)}(\vec{\mathrm{p}}_1{}' - \vec{\mathrm{p}}_1)\delta^{(3)}(\vec{\mathrm{p}}_2{}' - \vec{\mathrm{p}}_2)\delta_{\beta'\beta} \ ,$$

or in center-of-mass frame, where $\mathrm{E} = (\mathrm{u}_{1\beta} + \mathrm{u}_{2\beta})\mathrm{p}$, $\mathrm{u}_{1\beta} \equiv \mathrm{u_i}$, $\mathrm{u}_{2\beta} \equiv \mathrm{u_j}$

$$\langle \psi', \beta' \, |\psi, \beta \, \rangle$$
$$= (2\pi)^6 \cdot 4\mathrm{u}_{1\beta}\mathrm{u}_{2\beta}(\mathrm{u}_{1\beta} + \mathrm{u}_{2\beta}) \cdot \delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\delta^{(2)}(\hat{\vec{\mathrm{p}}}\,' - \hat{\vec{\mathrm{p}}})\delta_{\beta\beta'}.$$

- Two-particle state of definite angular momentum in the center-of-mass frame

$$|\mathrm{l}, \mathrm{m}, \mathcal{P}^{\mu}, \beta\rangle = \frac{1}{\sqrt{4\pi}} \int \mathrm{d}\hat{\vec{\mathrm{p}}}\; \mathrm{Y}_{\mathrm{l}}^{\mathrm{m}}(\hat{\vec{\mathrm{p}}})\, |\psi \ , \beta\rangle \, ,$$

so

$$\langle \mathrm{l}', \mathrm{m}', \mathcal{P}^{\mu\,\prime}, \beta' | \mathrm{l}, \mathrm{m}, \mathcal{P}^{\mu}, \beta\rangle$$
$$= \ 4\pi \mathrm{u}_{1\beta}\mathrm{u}_{2\beta}(\mathrm{u}_{1\beta} + \mathrm{u}_{2\beta}) \cdot (2\pi)^4 \delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\delta_{\mathrm{ll}'}\delta_{\mathrm{mm}'}\delta_{\beta\beta'}.$$

- Let us now write the T-matrix element

$$\mathrm{T}^{(\mathrm{l})}_{\mathrm{m}'\beta';\mathrm{m}\beta} = \langle \mathrm{l}, \mathrm{m}', \mathcal{P}^{\mu\,\prime}, \beta' | \mathrm{T} | \mathrm{l}, \mathrm{m}, \mathcal{P}^{\mu}, \beta \rangle \ .$$

$$\mathrm{T}^{(\mathrm{l})}_{\mathrm{m}'\beta';\mathrm{m}\beta} = \frac{1}{4\pi} \int \mathrm{d}\hat{\vec{\mathrm{p}}} \int \mathrm{d}\hat{\vec{\mathrm{p}}}\,' \; \mathrm{Y}^{\mathrm{m}}_{\mathrm{l}}(\hat{\vec{\mathrm{p}}}\,) \mathrm{Y}^{\mathrm{m}'\,*}_{\mathrm{l}}(\hat{\vec{\mathrm{p}}}\,') \, \langle \psi\,', \beta' | \mathrm{T} | \psi, \beta \rangle \ .$$

- After some calculations we arrive to

$$\mathrm{T}^{(\mathrm{l})}_{\mathrm{m}'\beta';\mathrm{m}\beta} = (2\pi)^4 \delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu}) \frac{\delta_{\mathrm{m}'\mathrm{m}}}{2} \int \mathrm{d}(\cos\gamma) \cdot \mathrm{P}_{\mathrm{l}}(\cos\gamma) \mathrm{M}_{\beta'\beta} \ .$$

- Finally, one defines the partial wave amplitude,

$$\mathrm{a}_{\mathrm{l},\beta'\beta} = \frac{1}{32\pi} \int \mathrm{d}(\cos\gamma) \cdot \mathrm{P}_{\mathrm{l}}(\cos\gamma) \mathrm{M}_{\beta'\beta},$$

and finds

$$\mathrm{T}^{(\mathrm{l})}_{\mathrm{m}'\beta';\mathrm{m}\beta} = 16\pi \cdot (2\pi)^4 \delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu}) \delta_{\mathrm{m}'\mathrm{m}} \; \mathrm{a}_{\mathrm{l},\beta'\beta} \ .$$

Unitarity relation and unitarity bounds for scalars with different sound speeds

Distinguishable particles

- Now we turn to the unitarity relation. Unitarity of S-matrix, $SS^\dagger = S^\dagger S = 1$ implies

$$T - T^\dagger = iTT^\dagger = iT^\dagger T\,.$$

- Inserting unit operator

$$\mathbb{1} = \int d^4\mathcal{P} \sum_{l,m,\beta} |l, m, \mathcal{P}^\mu, \beta\rangle \langle l, m, \mathcal{P}^\mu, \beta| \frac{1}{N(\beta)} + \ldots,$$

where where summation runs over all two-particle states, and

$$N(\beta) \equiv 2(2\pi)^5 u_{1\beta} u_{2\beta} (u_{1\beta} + u_{2\beta}).$$

in the right-hand side

$$-i\left(T^{(l)}_{m'\beta';m\beta} - T^{(l)\,*}_{m\beta;m'\beta'}\right) = \int d^4\mathcal{P}'' \sum_{m'',\beta''} \frac{1}{N(\beta'')} T^{(l)}_{m'\beta';m''\beta''} T^{(l)\,*}_{m\beta;m''\beta''}.$$

Unitarity bound

$$\mathrm{Im}\ \tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha} = \tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha}\tilde{\mathrm{a}}^*_{\mathrm{l},\alpha\alpha} + \sum_{\mathrm{M}} \mathrm{A}_{\mathrm{l},\alpha\mathrm{M}}\mathrm{A}^*_{\mathrm{l},\mathrm{M}\alpha}\ .$$

$$\mathrm{Im}\ \tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha} \geq |\tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha}|^2\ .$$

$$\left(\mathrm{Im}\ \tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha} - \frac{1}{2}\right)^2 + (\mathrm{Re}\ \tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha})^2 \leq \frac{1}{4}\ ,$$

$$|\mathrm{Re}\ \tilde{\mathrm{a}}_{\mathrm{l},\alpha\alpha}| \leq \frac{1}{2}, \tag{4}$$

Unitarity relation: example

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}_1^2 - \mathrm{u}_1^2(\vec{\nabla}\phi_1)^2\right) + \frac{1}{2}\left(\dot{\phi}_2^2 - \mathrm{u}_2^2(\vec{\nabla}\phi_2)^2\right) + \frac{\lambda_1}{4!}\phi_1^4 + \frac{\lambda_2}{4!}\phi_2^4 + \frac{\lambda_3}{4}\phi_1^2\phi_2^2\,, \tag{5}$$

where u_1 and u_2 are the two sound speeds.

$$\mathrm{Im}\ \mathrm{a}_{\mathrm{l},\alpha\beta} = \sum_\gamma \mathrm{g}_\gamma \mathrm{a}_{\mathrm{l},\alpha\gamma}\mathrm{a}^*_{\mathrm{l},\gamma\beta},$$

or in matrix form

$$\mathrm{Im}\ \mathrm{a}_\mathrm{l} = \sum_\gamma \mathrm{a}_\mathrm{l}\mathrm{g}\mathrm{a}_\mathrm{l}^\dagger, \tag{6}$$

Unitarity relation: example

$$\mathrm{M}_{\mathrm{tree}} = \begin{pmatrix} \mathrm{M}_{\alpha\alpha} & \mathrm{M}_{\alpha\beta} & \mathrm{M}_{\alpha\gamma} \\ \mathrm{M}_{\beta\alpha} & \mathrm{M}_{\beta\beta} & \mathrm{M}_{\beta\gamma} \\ \mathrm{M}_{\gamma\alpha} & \mathrm{M}_{\gamma\beta} & \mathrm{M}_{\gamma\gamma} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_2 \end{pmatrix}.$$

Unitarity relation: example