Recent Progress in Matrix Models: a brief review

Andrei Mironov

P.N.Lebedev Physics Institute,
IITP and ITEP at NRC “Kurchatov Institute"

Yerevan, 2023

Matrix models 30 year ago: basic properties

- Partition function of matrix models is a τ-function of integrable hierarchy
- It satisfies an infinite set of Ward identities forming Virasoro or W-algebras

Both properties survive (double scaling) continuum limits!

$\Downarrow$

They are suitable for describing universality classes

Ward identities are solvable.
This is because these universality classes describe topological theories, for instance, $2d$ gravity with matter = string theory.

Both properties are very general and are not related to particular matrix models.

New properties of matrix models

- Superintegrability
- W-representation of the partition function

The W-representation provides a connection with integrable many-body systems (like the rational Calogero system).

Both properties are related to concrete models, allow one to construct explicit solutions and are present in all known solvable examples.

Basic example: Hermitian one-matrix model

$$\int dH \exp\left[\,-\text{Tr}V(H)\,\right]$$

H is $N \times N$ Hermitian matrix.
$V(H)$ is a potential such that the integrals converges at suitable choices of integration contours.

Invariant correlators

$$\left\langle \prod_i \text{Tr}H^{k_i} \right\rangle = \int dH \exp\left[\,-\text{Tr}V(H)\,\right] \prod_i \text{Tr}H^{k_i}$$

Normalization: $\left\langle 1 \right\rangle = 1$.

Generating function of correlators

$$Z_N(t_k) = \left\langle \exp\left[\sum_k t_k \text{Tr}H^k\right]\right\rangle = \int dH \exp\left[-\text{Tr}V(H) + \sum_k t_k \text{Tr}H^k\right]$$

It is understood as a power series in parameters (sources) in t_k.

Basic example: Hermitian one-matrix model

$$\int dH \exp\left[\, -\mathrm{Tr} V(H)\, \right]$$

H is $N \times N$ Hermitian matrix.
$V(H)$ is a potential such that the integrals converges at suitable choices of integration contours.

Invariant correlators

$$\left\langle \prod_i \mathrm{Tr} H^{k_i} \right\rangle = \int dH \exp\left[\, -\mathrm{Tr} V(H)\, \right] \prod_i \mathrm{Tr} H^{k_i}$$

Normalization: $\left\langle 1 \right\rangle = 1$.

Generating function of correlators

$$Z_N(t_k) = \left\langle \exp\left[\sum_k t_k \mathrm{Tr} H^k \right] \right\rangle = \int dH \exp\left[-\mathrm{Tr} V(H) + \sum_k t_k \mathrm{Tr} H^k \right]$$

It is understood as a power series in parameters (sources) in t_k.

Basic example: Hermitian one-matrix model

$$\int dH \exp\left[\, -\mathrm{Tr} V(H)\, \right]$$

H is $N \times N$ Hermitian matrix.
$V(H)$ is a potential such that the integrals converges at suitable choices of integration contours.

Invariant correlators

$$\left\langle \prod_i \mathrm{Tr} H^{k_i} \right\rangle = \int dH \exp\left[\, -\mathrm{Tr} V(H)\, \right] \prod_i \mathrm{Tr} H^{k_i}$$

Normalization: $\left\langle 1 \right\rangle = 1$.

Generating function of correlators

$$Z_N(t_k) = \left\langle \exp\left[\sum_k t_k \mathrm{Tr} H^k \right] \right\rangle = \int dH \exp\left[-\mathrm{Tr} V(H) + \sum_k t_k \mathrm{Tr} H^k \right]$$

It is understood as a power series in parameters (sources) in t_k.

Integrability

Key property: determinant representation:

$$Z_N(t_k) = \det_{1\le i,j\le N} C_{i+j-2}$$

Moment matrix

$$C_n = \int dh h^n \exp\left[-V(h) + \sum_k t_k h^k \right]$$

$Z_N(t_k)$ is a τ-function of the Toda chain hierarchy. t_k are time variables of the hierarchy.

Define

$$e^{\varphi_N} = \frac{Z_{N+1}(t_k)}{Z_N(t_k)}$$

Then

$$\frac{\partial^2 \varphi_N}{\partial t_1^2} = e^{\varphi_{N+1}-\varphi_N} - e^{\varphi_N - \varphi_{N-1}}$$

Integrability

Key property: determinant representation:

$$Z_N(t_k) = \det_{1\le i,j\le N} C_{i+j-2}$$

Moment matrix

$$C_n = \int dh h^n \exp\left[-V(h) + \sum_k t_k h^k \right]$$

$Z_N(t_k)$ is a τ-function of the Toda chain hierarchy. t_k are time variables of the hierarchy.

Define

$$e^{\varphi_N} = \frac{Z_{N+1}(t_k)}{Z_N(t_k)}$$

Then

$$\frac{\partial^2 \varphi_N}{\partial t_1^2} = e^{\varphi_{N+1}-\varphi_N} - e^{\varphi_N - \varphi_{N-1}}$$

Integrability

Key property: determinant representation:

$$Z_N(t_k) = \det_{1\le i,j\le N} C_{i+j-2}$$

Moment matrix

$$C_n = \int dh h^n \exp\left[-V(h) + \sum_k t_k h^k \right]$$

$Z_N(t_k)$ is a τ-function of the Toda chain hierarchy. t_k are time variables of the hierarchy.

Define

$$e^{\varphi_N} = \frac{Z_{N+1}(t_k)}{Z_N(t_k)}$$

Then

$$\frac{\partial^2 \varphi_N}{\partial t_1^2} = e^{\varphi_{N+1}-\varphi_N} - e^{\varphi_N - \varphi_{N-1}}$$

Ward identities=Virasoro constraints

1) Make a redefinition of parameters t_k's (for a polynomial $V(H)$):

$$-\text{Tr}V(H) + \sum_k t_k \text{Tr} H^k \longrightarrow \sum_k t_k \text{Tr} H^k$$

2) Note that integral of full derivative is zero:

$$\int dH \text{Tr} \frac{\partial}{\partial H} \left(H^{n+1} \exp \left[\sum_k t_k \text{Tr} H^k \right] \right) = 0$$

It is written in the form

$$\underbrace{\left(\sum_k k t_k \frac{\partial}{\partial t_{k+n}} + \sum^{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b} \right)}_{L_n} \int dH \exp \left[\sum_k t_k \text{Tr} H^k \right] = 0$$

$$[L_n, L_m] = (n-m) L_{n+m} \qquad - \quad \text{Virasoro algebra}$$

Ward identities=Virasoro constraints

1) Make a redefinition of parameters t_k's (for a polynomial $V(H)$):

$$-\text{Tr}V(H) + \sum_k t_k \text{Tr} H^k \longrightarrow \sum_k t_k \text{Tr} H^k$$

2) Note that integral of full derivative is zero:

$$\int dH \, \text{Tr} \frac{\partial}{\partial H} \left(H^{n+1} \exp\left[\sum_k t_k \text{Tr} H^k \right] \right) = 0$$

It is written in the form

$$\underbrace{\left(\sum_k k t_k \frac{\partial}{\partial t_{k+n}} + \sum^{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b} \right)}_{L_n} \int dH \exp\left[\sum_k t_k \text{Tr} H^k \right] = 0$$

$$[L_n, L_m] = (n-m) L_{n+m} \qquad - \quad \text{Virasoro algebra}$$

Ward identities=Virasoro constraints

1) Make a redefinition of parameters t_k's (for a polynomial $V(H)$):

$$-\text{Tr}V(H) + \sum_k t_k \text{Tr} H^k \longrightarrow \sum_k t_k \text{Tr} H^k$$

2) Note that integral of full derivative is zero:

$$\int dH \text{Tr} \frac{\partial}{\partial H}\left(H^{n+1} \exp\left[\sum_k t_k \text{Tr} H^k \right]\right) = 0$$

It is written in the form

$$\underbrace{\left(\sum_k k t_k \frac{\partial}{\partial t_{k+n}} + \sum^{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b}\right)}_{L_n} \int dH \exp\left[\sum_k t_k \text{Tr} H^k \right] = 0$$

$$[L_n, L_m] = (n-m) L_{n+m} \qquad - \quad \text{Virasoro algebra}$$

Solutions of matrix models

- Matrix model integrable hierarchies have many solutions.
- The Ward identities for non-Gaussian $V(H)$ have many solutions.

Example of $V(H) = H^3$ (Dijkgraaf-Vafa solution) gives rise to the Airy type integrals

$$\int dh e^{h^3+\dots}$$

There are two choices of integration contours C_1 and C_2.
The partition function is labeled by N_1 and N_2 $(N_1 + N_2 = N)$.

New results in matrix models are associated with concrete choices of $V(H)$ and integration contours. They provide a procedure of solving with these data fixed.

Solutions of matrix models

- Matrix model integrable hierarchies have many solutions.
- The Ward identities for non-Gaussian $V(H)$ have many solutions.

Example of $V(H) = H^3$ (Dijkgraaf-Vafa solution) gives rise to the Airy type integrals

$$\int dh e^{h^3 + \dots}$$

There are two choices of integration contours C_1 and C_2.
The partition function is labeled by N_1 and N_2 ($N_1 + N_2 = N$).

New results in matrix models are associated with concrete choices of $V(H)$ and integration contours. They provide a procedure of solving with these data fixed.

Solutions of matrix models

- Matrix model integrable hierarchies have many solutions.
- The Ward identities for non-Gaussian $V(H)$ have many solutions.

Example of $V(H) = H^3$ (Dijkgraaf-Vafa solution) gives rise to the Airy type integrals

$$\int dh e^{h^3 + \ldots}$$

There are two choices of integration contours C_1 and C_2.
The partition function is labeled by N_1 and N_2 ($N_1 + N_2 = N$).

New results in matrix models are associated with concrete choices of $V(H)$ and integration contours. They provide a procedure of solving with these data fixed.

Superintegrability

In classical mechanics:

- Integrability: there are N integrals of motion in involution

$\Downarrow$ Liouville theorem

The equations of motion are solved in quadratures

- Superintegrability: there are more than N integrals of motion. Usually, it gives rise to explicit solutions.

An example: in the Coulomb system, there is an additional integral, the well-known Laplace-Runge-Lenz vector (the potential is $V = -\frac{g}{r}$, $\vec{L}$ is the angular momentum vector),

$$\vec{p} \times \vec{L} - gm\vec{r}$$

Superintegrability in matrix models:

For many/all (?) potentials, there is a basis that admits explicit expressions for arbitrary correlators. There is a hidden symmetry similarly to the Coulomb system case.

Superintegrability

In classical mechanics:

- Integrability: there are N integrals of motion in involution

$\Downarrow$ Liouville theorem

The equations of motion are solved in quadratures

- Superintegrability: there are more than N integrals of motion. Usually, it gives rise to explicit solutions.

An example: in the Coulomb system, there is an additional integral, the well-known Laplace-Runge-Lenz vector (the potential is $V = -\frac{g}{r}$, $\vec{L}$ is the angular momentum vector),

$$\vec{p} \times \vec{L} - gm\vec{r}$$

Superintegrability in matrix models:

For many/all (?) potentials, there is a basis that admits explicit expressions for arbitrary correlators. There is a hidden symmetry similarly to the Coulomb system case.

Gaussian Hermitian one-matrix model: $V(H) = -\frac{1}{2}H^2$

The basis is given by Schur functions.

The Schur functions S_R are symmetric polynomials of eigenvalues of the matrix H, or graded polynomials of $\mathrm{Tr}H^k$. They form a complete basis in the space of graded invariant polynomials. They are labeled by partitions, or Young diagrams R: $R_1 \geq R_2 \geq \ldots \geq R_l > 0$. Examples:

$$S_\emptyset = 1$$

$$S_{[1]} = \mathrm{Tr}H$$

$$S_{[2]} = \frac{(\mathrm{Tr}H)^2}{2} + \frac{\mathrm{Tr}H^2}{2}$$

$$S_{[1,1]} = \frac{(\mathrm{Tr}H)^2}{2} - \frac{\mathrm{Tr}H^2}{2}$$

General formula:

$$\exp\left[\sum_k \frac{p_k}{k} z^k\right] = \sum_i h_i(p_k) z^i$$

$$S_R(p_k) = \det_{i,j} h_{R_i - i + j}(p_k) \qquad\qquad p_k \longrightarrow \mathrm{Tr}H^k$$

Superintegrability in the Gaussian model

Formula for the general correlator:

$$\left\langle S_R(\mathrm{Tr}H^k)\right\rangle = \frac{S_R(p_k = N)S_R(p_k = \delta_{k,2})}{S_R(p_k = \delta_{k,1})}$$

The point is that the Schur function S_R is the character of the linear group $GL(N)$ in the representation labeled by the Young diagram R. Hence, averages of characters are proportional to characters at peculiar points.

The partition functions is

$$Z_N(t_k) = \sum_R \frac{S_R(p_k = N)S_R(p_k = \delta_{k,2})}{S_R(p_k = \delta_{k,1})} S_R(p_k = kt_k)$$

due to the Cauchy identity:

$$\exp\left[\sum_k t_k \mathrm{Tr}H^k\right] = \sum_R S_R(\mathrm{Tr}H^k)S_R(p_k = kt_k)$$

More examples of superintegrability

Various other examples of superintegrable models. Choosing non-Gaussian potentials:

- Hermitian matrix model with monomial $V(H)$. The basis is still given by the Schur polynomials

$$\Big\langle S_R \Big\rangle_a = \int_{C_a} S_R(\mathrm{Tr} H^k)\cdot e^{-\frac{1}{s}\mathrm{tr}\, H^s} dH = S_R(p_k=\delta_{k,s})\cdot \prod_{(\alpha,\beta)\in R} [[N+\alpha-\beta]]_{s,0}\cdot[[N+\alpha-\beta]]_{s,a}$$

  for $N=0$ or $=a$ mod s, and $[[n]]_{s,a}=n$ if $n=a$ mod s and $=1$ otherwise.

- Logarithmic potentials:

$$\Big\langle S_R(\mathrm{Tr} H^k)\Big\rangle = \int S_R(\mathrm{Tr} H^k)\exp\Big[u\mathrm{Tr}\log H + v\mathrm{Tr}\log(1-H)\Big]dH = \frac{S_R(p_k=N)\cdot S_R(p_k=u+N)}{S_R(p_k=u+v+2N)}$$

- Potential that is a square of logarithm:

$$\Big\langle S_R\Big\rangle := \int S_R(\mathrm{Tr} H^k)\exp\left[-\frac{\mathrm{Tr}(\log H)^2}{2g^2}\right]dH = A^{|R|}q^{2\varkappa_R}\cdot S_R\left(p_k=\frac{A^k-A^{-k}}{q^k-q^{-k}}\right)$$

  where $\varkappa_R=\sum_{(\mu,\nu)\in R}(\mu-\nu)$, $q=\exp\left(\frac{g^2}{2}\right)$ and $A=q^N$.

Examples of models with the same superintegrability properties:

- Models depending on the external matrix. One example is

$$\int S_R(\mathrm{Tr}H^k)\exp\left[-\frac{1}{2}\mathrm{Tr}AHAH\right]dH = \frac{S_R(p_k=\mathrm{Tr}A^{-k})\cdot S_R(p_k=\delta_{k,2})}{S_R(p_k=\delta_{k,1}}$$

  Here the basis is still given by the Schur functions.
  Another example is the generalized Kontsevich models. The basis in the second case is given by the Hall-Littlewood polynomials at special values of the parameter.

- Complex matrix model instead of the Hermitian one. The basis is still given by the Schur polynomials:

$$\int S_R(\mathrm{Tr}(ZZ^\dagger)^k)\exp\left[-\mathrm{Tr}AZBZ^\dagger\right]d^2Z = \frac{S_R(p_k=\mathrm{Tr}A^{-k})\cdot S_R(p_k=\mathrm{Tr}B^{-k})}{S_R(p_k=\delta_{k,1}}$$

- Models of orthogonal and real matrices, and, generally, the β-ensemble instead of matrix model. The basis is given by the Jack polynomials. There is also further deformation to the (q,t)-matrix models and the Macdonald polynomials.
- Tensor models instead of matrix models. The basis is given by the generalized characters.

W-representation

The second newly discovered property of the matrix models is their W-representation. In the Gaussian Hermitian matrix model case, it is

$$\int dH \exp\left[-\frac{1}{2}\mathrm{Tr}H^2 + \sum_k t_k \mathrm{Tr}H^k\right] = e^{\frac{1}{2}\hat{W}_2}\cdot 1$$

with

$$\hat{W}_2 = \sum_k klt_kt_l\frac{\partial}{\partial t_{k+l-2}} + \sum_{k,l}(k+l+2)t_{k+l+2}\frac{\partial^2}{\partial t_k\partial t_l} + 2N\sum_k(k+2)t_{k+2}\frac{\partial}{\partial t_k} + 2N^2t_2 + Nt_1^2$$

One can see that

$$e^{\frac{1}{2}\hat{W}_2}\cdot 1 = \sum_R \frac{S_R(p_k=N)S_R(p_k=\delta_{k,2})}{S_R(p_k=\delta_{k,1})}S_R(p_k=kt_k)$$

In the variables q_i: $t_k = \frac{1}{k}\sum_i^N q_i$, one obtains the rational Calogero Hamiltonian at the free fermion point:

$$\hat{W}_2 = \sum_i \frac{\partial^2}{\partial q_i^2} + 2\sum_{i\neq j}\frac{1}{q_i-q_j}\frac{\partial}{\partial q_i}$$

Infinite commutative family of W-operators

There is an infinite commutative family of $\hat{W}_p$:

$$[\hat{W}_p, \hat{W}_{p'}] = 0$$

These operators are elements of the $W_{1+\infty}$ algebra.

They induce a two-matrix model generalization:

$$Z_N(t_k, \bar{t}_k) = e^{\sum_k \bar{t}_k \hat{W}_k} \cdot 1 = \int dX dY \exp\left[\mathrm{Tr}XY + \sum_k \left(t_k \mathrm{Tr}X^k + \bar{t}_k \mathrm{Tr}Y^k\right)\right]$$

where X is Hermitian matrix and Y is anti-Hermitian.
The superintegrability gives rise to the expansion

$$Z_N(t_k, \bar{t}_k) = \sum_R \frac{S_R(p_k = N) S_R(p_k = k\bar{t}_k) S_R(p_k = kt_k)}{S_R(p_k = \delta_{k,1})}$$

The reduction to the Gaussian case is immediate: $\bar{t}_k = \frac{1}{2}\delta_{k,2}$.

Infinite commutative family of W-operators

There is an infinite commutative family of $\hat{W}_p$:

$$[\hat{W}_p, \hat{W}_{p'}] = 0$$

These operators are elements of the $W_{1+\infty}$ algebra.

They induce a two-matrix model generalization:

$$Z_N(t_k, \bar{t}_k) = e^{\sum_k \bar{t}_k \hat{W}_k} \cdot 1 = \int dX dY \exp\left[\mathrm{Tr} XY + \sum_k \left(t_k \mathrm{Tr} X^k + \bar{t}_k \mathrm{Tr} Y^k\right)\right]$$

where X is Hermitian matrix and Y is anti-Hermitian.
The superintegrability gives rise to the expansion

$$Z_N(t_k, \bar{t}_k) = \sum_R \frac{S_R(p_k = N) S_R(p_k = k\bar{t}_k) S_R(p_k = kt_k)}{S_R(p_k = \delta_{k,1})}$$

The reduction to the Gaussian case is immediate: $\bar{t}_k = \frac{1}{2}\delta_{k,2}$.

Infinite commutative family of W-operators

There is an infinite commutative family of $\hat{W}_p$:

$$[\hat{W}_p, \hat{W}_{p'}] = 0$$

These operators are elements of the $W_{1+\infty}$ algebra.

They induce a two-matrix model generalization:

$$Z_N(t_k, \bar{t}_k) = e^{\sum_k \bar{t}_k \hat{W}_k} \cdot 1 = \int dX dY \exp\left[\mathrm{Tr} XY + \sum_k \left(t_k \mathrm{Tr} X^k + \bar{t}_k \mathrm{Tr} Y^k\right)\right]$$

where X is Hermitian matrix and Y is anti-Hermitian.
The superintegrability gives rise to the expansion

$$Z_N(t_k, \bar{t}_k) = \sum_R \frac{S_R(p_k = N) S_R(p_k = k\bar{t}_k) S_R(p_k = k t_k)}{S_R(p_k = \delta_{k,1})}$$

The reduction to the Gaussian case is immediate: $\bar{t}_k = \frac{1}{2}\delta_{k,2}$.

Rational Calogero many-body system

Rational Calogero many-body system is superintegrable, and its Hamiltonians are just $\hat{W}_k$!

How to introduce an arbitrary Calogero coupling? Integration over angular variables:

$$\int dH \exp\left[-\mathrm{Tr}V(H) + \sum_k t_k \mathrm{Tr}H^k\right] \sim \int \Delta(x)^2 \prod_i^N \exp\left[-V(x_i) + \sum_k t_k x_i^k\right] dx_i$$

where x_i are the eigenvalues of the matrix H, $\Delta(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant.

β-ensemble:

$$Z_N^{(\beta)}(t_k) = \int \Delta(x)^{2\beta} \prod_i^N \exp\left[-V(x_i) + \sum_k t_k x_i^k\right] dx_i$$

The Gaussian β-ensemble with $V(x) = \frac{1}{2}x^2$ has the W-representation

$$Z_N^{(\beta)}(t_k) = e^{\frac{1}{2}\hat{W}_2^{(\beta)}} \cdot 1$$

and, once again in variables q_i, one obtains the rational Calogero Hamiltonian at arbitrary coupling

$$\Delta(q)^{\beta}\hat{W}_2^{(\beta)}\Delta(q)^{-\beta} = \sum_i \frac{\partial^2}{\partial q_i^2} - 2\beta(\beta-1)\sum_{i\neq j}\frac{1}{(q_i-q_j)^2}$$

Rational Calogero many-body system

Rational Calogero many-body system is superintegrable, and its Hamiltonians are just $\hat{W}_k$!

How to introduce an arbitrary Calogero coupling? Integration over angular variables:

$$\int dH \exp\left[-\mathrm{Tr}V(H)+\sum_k t_k \mathrm{Tr}H^k\right] \sim \int \Delta(x)^2 \prod_i^N \exp\left[-V(x_i)+\sum_k t_k x_i^k\right] dx_i$$

where x_i are the eigenvalues of the matrix H, $\Delta(x)=\prod_{i<j}(x_i-x_j)$ is the Vandermonde determinant.

β-ensemble:

$$Z_N^{(\beta)}(t_k)=\int \Delta(x)^{2\beta} \prod_i^N \exp\left[-V(x_i)+\sum_k t_k x_i^k\right] dx_i$$

The Gaussian β-ensemble with $V(x)=\frac{1}{2}x^2$ has the W-representation

$$Z_N^{(\beta)}(t_k)=e^{\frac{1}{2}\hat{W}_2^{(\beta)}}\cdot 1$$

and, once again in variables q_i, one obtains the rational Calogero Hamiltonian at arbitrary coupling

$$\Delta(q)^\beta \hat{W}_2^{(\beta)} \Delta(q)^{-\beta}=\sum_i \frac{\partial^2}{\partial q_i^2}-2\beta(\beta-1)\sum_{i\neq j}\frac{1}{(q_i-q_j)^2}$$

Rational Calogero many-body system is superintegrable, and its Hamiltonians are just $\hat{W}_k$!

How to introduce an arbitrary Calogero coupling? Integration over angular variables:

$$\int dH \exp\left[-\mathrm{Tr}V(H)+\sum_k t_k \mathrm{Tr}H^k\right] \sim \int \Delta(x)^2 \prod_i^N \exp\left[-V(x_i)+\sum_k t_k x_i^k\right] dx_i$$

where x_i are the eigenvalues of the matrix H, $\Delta(x)=\prod_{i<j}(x_i-x_j)$ is the Vandermonde determinant.

β-ensemble:

$$Z_N^{(\beta)}(t_k) = \int \Delta(x)^{2\beta} \prod_i^N \exp\left[-V(x_i)+\sum_k t_k x_i^k\right] dx_i$$

The Gaussian β-ensemble with $V(x)=\frac{1}{2}x^2$ has the W-representation

$$Z_N^{(\beta)}(t_k) = e^{\frac{1}{2}\hat{W}_2^{(\beta)}} \cdot 1$$

and, once again in variables q_i, one obtains the rational Calogero Hamiltonian at arbitrary coupling

$$\Delta(q)^\beta \hat{W}_2^{(\beta)} \Delta(q)^{-\beta} = \sum_i \frac{\partial^2}{\partial q_i^2} - 2\beta(\beta-1)\sum_{i\neq j}\frac{1}{(q_i-q_j)^2}$$

Thank you for your attention!