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Instanton Calculus, Spectral Flow and Beta Functions

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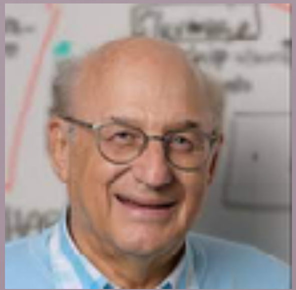


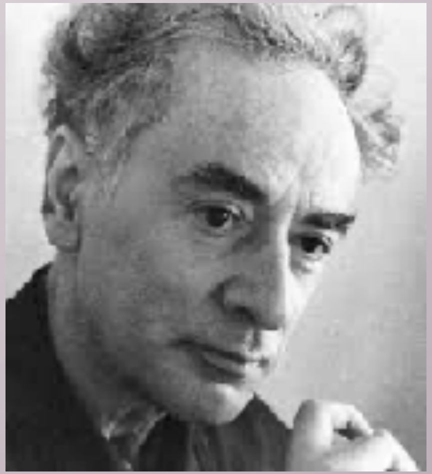
Nor-Amberd 198?

Photo by Edward Shuryak

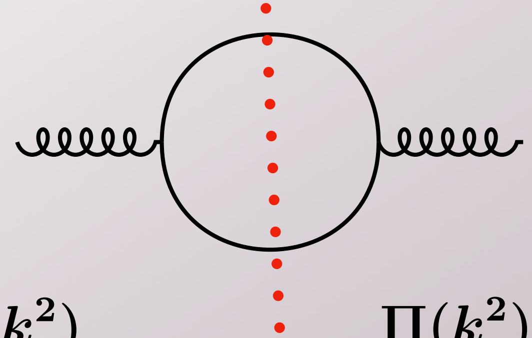
Introduction

Our goal is to clarify a certain confusion regarding calculations of the β function by performing the path integral around instantons—or more generally, classical solutions— in theories supporting them. The confusion arises from a specific relationship between the number of zero modes and the asymptotically free contribution to the β function. To provide more details, we present a brief introduction.

- L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, "An asymptotic expression for the electron green function in quantum electrodynamics," Dokl.Akad.Nauk SSSR 95 (1954) 773.



Landau and his collaborators provided a general explanation for why all field theories known at that time were infrared-free.



$$\Pi_{\mu\nu} = (k_\mu k_\nu - g_{\mu\nu} k^2)\Pi(k^2)$$

$$\Pi(k^2) = \frac{1}{\pi}\int_{4m^2}^{M_0^2} \frac{ds}{s-k^2}\,\mathrm{Im}\Pi(s)$$

$$e^2(k^2) = e_0^2\big(1 - e_0^2\Pi(k^2)\big)$$

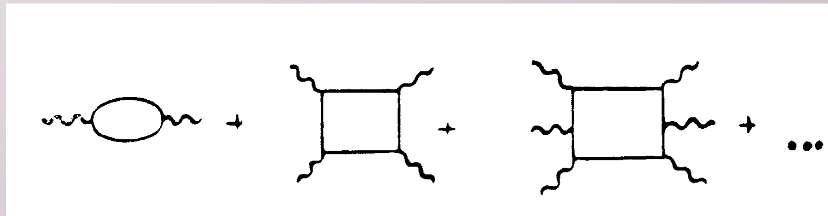
$$\mathrm{Im}\Pi(s) = \frac{\sigma_{\mathrm{annih}}}{4\pi\alpha_0^2/3s}$$

Screening of the charge, leading to the zero charge problem, looked generic.

The first counterexample in frameworks of 1+1 d theory was provided by Alexei Anselm in 1959.

A. Anselm "Field model with a nonvanishing renormalized charge" ZhETF 36 (1959) 363

In1965, Vanyashin and Terent'ev studied electrodynamics of massive vector field. Found antisreening and asymptotic freedom behavior.



Vladimir Vanyashin

Mikhail Terent'ev
1935-1996

SOVIET PHYSICS JETP VOLUME 21, NUMBER 2 AUGUST, 1965

THE VACUUM POLARIZATION OF A CHARGED VECTOR FIELD

V. S. VANYASHIN and M. V. TERENT'EV

Submitted to JETP editor June 13, 1964; resubmitted October 10, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 565-573 (February, 1965)

The nonlinear additions to the Lagrangian of a constant electromagnetic field, caused by the vacuum polarization of a charged vector field, are calculated in the special case in which the gyromagnetic ratio of the vector boson is equal to 2. The result is exact for an arbritrarily strong electromagnetic field, but does not take into account radiative corrections, which can play an important part in the unrenormalized electrodynamics of a vector boson. The anomalous character of the charge renormalization is pointed out.

Pre QCD: Khriplovich's derivation of antiscreening in Yang-Mills, 1969

Yulik Khriplovich,

My mentor, colleague and friend

SOVIET JOURNAL OF NUCLEAR PHYSICS VOLUME 10, NUMBER 2 FEBRUARY, 1970

GREEN'S FUNCTIONS IN THEORIES WITH A NON-ABELIAN GAUGE GROUP

I. B. KHRIPLOVICH

Institute for Nuclear Physics, Siberian Section USSR Academy of Sciences

Submitted December 21, 1968

Yad. Fiz. 10, 409–424 (August, 1969)

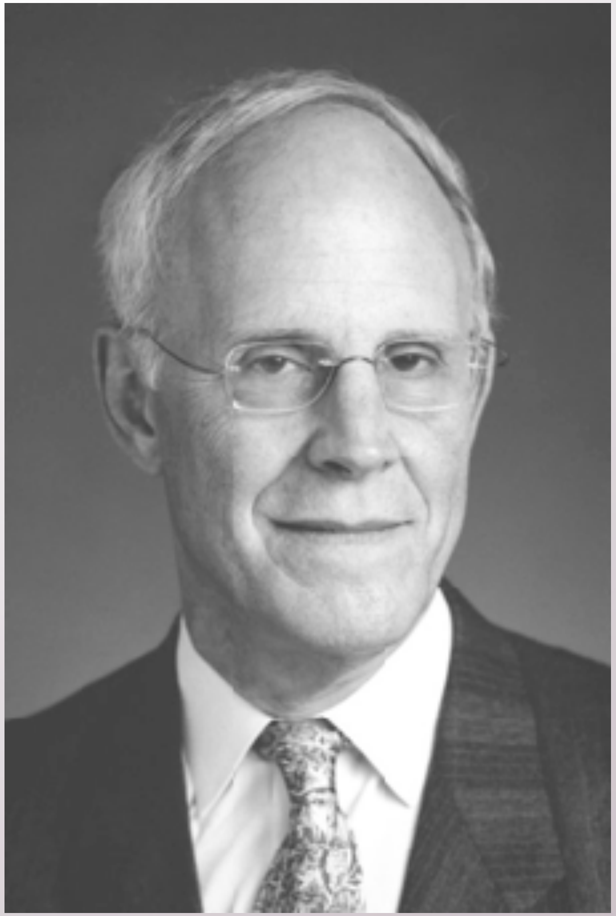
SU(2) Yang-Mills in the radiation (Coulomb) gauge.

The Green function for non-Abelian gauge field

$$\int d^4x \exp(-ipx)\langle A_\mu^\alpha(x) A_\nu^\beta(0)\rangle = i\delta^{\alpha\beta} D_{\mu\nu}(p)$$

$$D_{00} = -\frac{1}{\vec{p}^2}\left\{1 + \frac{g^2}{4\pi^2}\left[8\ln\frac{\Lambda_1^2}{\vec{p}^2} - \frac{2}{3}\ln\frac{\Lambda^2}{-p^2}\right]\right\}$$

Non-dispersive part produced anti-screening and dominates numerically (12 times larger)

50 years of QCD, celebrated in September at UCLA, is counted from their 1973 papers.

They got Nobel Prize in 2004.

We saw that AF is associated with non-dispersive contributions in the unitary Coulomb gauge.
But it also simple to see it in covariant gauges where ghost fields appear.

Split SU(2) gauge field as

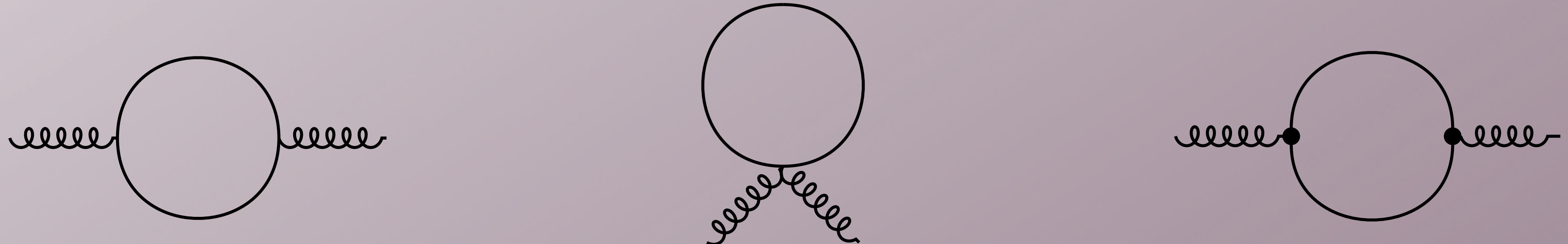
$$A^a_\mu = \mathcal{A}^a_\mu + a^a_\mu$$

to the background $\mathcal{A}^a_\mu$ and quantum a^a_μ. Fix the gauge

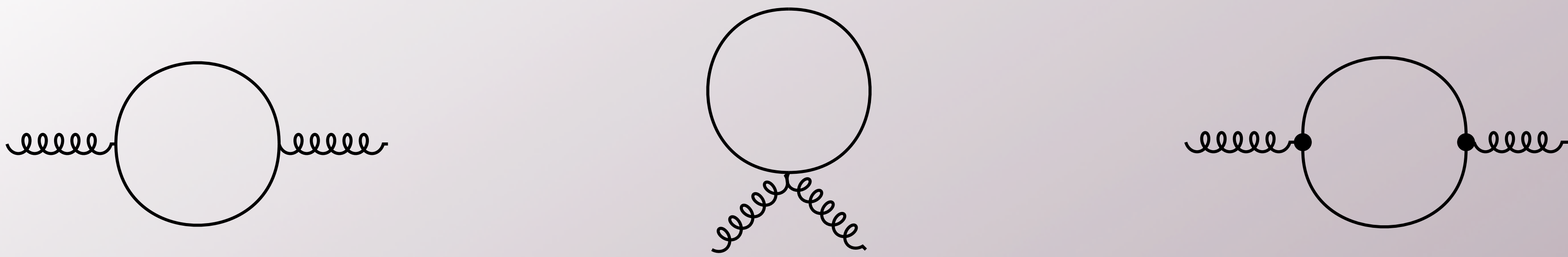
$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2g_0^2}\left(\mathcal{D}^{ab}_\mu a^{b\mu}\right)^2 \qquad \mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu + f^{acb}\mathcal{A}^c_\mu$$

together with adding the Faddeev-Popov ghost fields. Altogether,

$$\mathcal{L}_2 = \frac{1}{2g_0^2}\, a^a_\mu \left[\eta^{\mu\nu}(\mathcal{D}_\gamma\mathcal{D}^\gamma)^{ab} + 2f^{acb}\mathcal{F}^c_{\mu\nu}\right] a^b_\nu - \frac{1}{2g_0^2}\,\bar{c}^a(\mathcal{D}_\gamma\mathcal{D}^\gamma)^{ab}c^b$$



(a) (b) (c)



(a) (b) (c)

$$
\begin{aligned}
\mathcal{L}_{eff} &= -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{\mu\nu\,a}\left[\frac{1}{g_0^2}+\frac{1}{8\pi^2}\left(-8+\frac{4}{3}\right)\log\frac{M}{\mu}-\frac{1}{8\pi^2}\frac{2}{3}\log\frac{M}{\mu}\right] \\
&= -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{\mu\nu a}\frac{1}{8\pi^2}\left[\frac{8\pi^2}{g_0^2}-8\log\frac{M}{\mu}+\frac{2}{3}\log\frac{M}{\mu}\right],
\end{aligned}
$$

AF contribution is due to the magnetic spin interaction while “electric” interaction of gauge field together with ghosts gives IRF term which is 12 times smaller.

Here, in Yerevan Physics Institute, in 1977 it was George Savvidy who in his works and together with Sergei Matinyan discovered instability of YM at large external fields suggesting an existence of the gluon condensate.

We used the notion of gluon condensate in 1979, introducing SVZ sum rules, but missed at that time to acknowledgethe above papers. I'd like to repeat now my belated apologies.

George Savvidy

Sergei Matinyan
1931-2017

Large fields appear in the Belavin-Polyakov-Schwarz-Tyupkin instanton solution in 1975. Instanton measure, accounting for quantum fluctuations, was calculated by Gerard 't Hooft in1978.

$$d\mu_{\text{inst}} = \text{const} \times \int \frac{d^4x_0\, d\rho}{\rho^5} (M\rho)^8 \left(\frac{8\pi^2}{g_0^2}\right)^4 \exp\left(-\frac{8\pi^2}{g_0^2} + \Delta_{\text{gl}} + \Delta_{\text{gh}}\right)$$

Pre-exponential factors come from 8 zero modes. The *bona fide* quantum corrections in the instanton background, which take into account only non-zero modes are

$$\Delta_{\text{gl}} + \Delta_{\text{gh}} = -\frac{2}{3} \log M\rho$$

Then,

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g_0^2} - 8 \log M\rho + \frac{2}{3} \log M\rho$$

Anti-screening contributions coming from zero modes coincide with non-dispersive terms in the Coulomb gauge. Note an absence of non-zero modes in SUSY.

Taking at face value, this pecularity becomes a source of confusion. It is tempting to generalize this to other classical solutions. Although two-dimensional CP(1) present a similar situation, there is no reason that it works in general. In fact, even three-dimensional YM fails this test.

Specific of four-dimensional YM was discovered by 't Hooft who showed that non-zero eigenvalues for instanton fluctuation modes do not depend on spin. We do not know what is deep reason for this spin independence.

Setup

We are studying an O(N) invariant Euclidean scalar field theory in d = 4 dimensions

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 - \frac{g_0}{4!}(\phi_a^2)^2, \quad a = 1, \dots, N, \quad g_0 > 0.$$

Note that the chosen sign of coupling corresponds to classically unstable theory. Despite the negative potential, this theory is well-defined perturbatively, meaning that there are no instabilities arising at perturbative level.

At the same time such a non-standard choice of the coupling leads to an interesting feature of the theory, it is asymptotically free Symanzik '75?

The effective potential as a way to calculate β-function.
Expending near $\phi_N = \phi_0$,

$$\phi_N = \phi_0 + \varphi_N, \quad \phi_a = \varphi_a, \quad a \neq N,$$

We get

$$S = S_0 + \int d^4x \left[\frac{1}{2}(\partial\varphi_N)^2 + \frac{1}{2}(\partial\varphi_a)^2 - \frac{g_0}{4}\,\phi_0^2\varphi_N^2 - \frac{g_0}{12}\,\phi_0^2\varphi_a^2 \right], \quad a \neq N.$$

and the effective action

$$\Gamma[\phi_0] = S_0 + \frac{1}{2}\,\mathrm{Tr}\log\left[-\partial^2 - \frac{g_0\phi_0^2}{2}\right] + \frac{N-1}{2}\,\mathrm{Tr}\log\left[-\partial^2 - \frac{g_0\phi_0^2}{6}\right].$$

For constant $\phi_0 = \phi_c$ and Pauli-Villars regularization we come to

$$\Gamma_R[\phi_c] = \int d^4x \left[-\frac{g_0\phi_c^4}{4!}\left(1 + \frac{g_0}{32\pi^2}\frac{N+8}{3}\log\frac{M}{\mu}\right) + b_4M^4 + b_2M^2g\phi_c^2 \right]$$

Here the normalization point μ represents the lower limit in momentum integration and $\mu \gg g\phi_c^2$.

Running coupling shows AF regime.

$$g(\mu) = g_0 \left(1 + \frac{g_0}{16\pi^2} \frac{N+8}{3} \log \frac{M}{\mu}\right)$$

$$\beta_g(g) = -\frac{g^2}{16\pi^2} \frac{N+8}{3}$$

Fubini - Lipatov instanton

In '76 and '77 they found a classical solution

$$\phi_N = \phi_{FL} \equiv 4\sqrt{\frac{3}{g_0}} \frac{\rho}{r^2+\rho^2}, \quad \phi_i = 0, \quad i \neq N,$$

which is called Fubini-Lipatov instanton (actually a bounce). Path integral near this configuration,

$$I_{FL} = \int \mathcal{D}\varphi e^{-S[\phi_{FL}+\varphi]},$$

contains some number of zero mode integrations.

Five zero modes are due to the broken space-time symmetries: 4 translations plus dilation. Additionally $N-1$ modes due to internal symmetry breaking: $SO(N) \to SO(N-1)$. Furthermore, one negative mode with the eigenvalue λ_-. Thus,

$$I_{FL} = \int \sqrt{\frac{(M^2+\lambda_-)}{\lambda_-}}\, dX\, M^{(N+4)} \exp\{-\Gamma_R[\phi_{FL}]\}$$

where M denotes Pauli-Villars regulator and

$$\Gamma_R[\phi_{FL}] = \frac{16\pi^2}{g_0} + \text{nonzero modes}$$

It means that

$$\frac{16\pi^2}{g(\rho)} = \frac{16\pi^2}{g_0} - (N+5)\log M\rho + \text{nonzero modes logs}$$

Compare with perturbative

$$\frac{16\pi^2}{g(\mu)} = \frac{16\pi^2}{g_0} - \frac{N+8}{3}\log\frac{M}{\mu}$$

Discussion

Mapping on a sphere to discretize levels.

Let take $N = 1$

$$\mathcal{P}_{FL} = \int dX\, M^6 \left[\frac{\det'(\mathcal{M}_{\mathcal{I}} + M^2)}{\det' \mathcal{M}_{\mathcal{I}}} \frac{\det \mathcal{M}_0}{\det(\mathcal{M}_0 + M^2)}\right]^{1/2}$$

From nonzero modes

$$\mathcal{R}_{FL} = M^6 \left[\frac{\det'(\mathcal{M}_{\mathcal{I}} + M^2)}{\det(\mathcal{M}_0 + M^2)}\right]^{1/2}$$

$$\frac{\det'(\mathcal{M}_{\mathcal{I}} + M^2)}{\det(\mathcal{M}_0 + M^2)} = \frac{\prod_{\ell=2}^{\Lambda} (\lambda_\ell^{\mathcal{I}} + M^2)^{\nu_\ell/2}}{\prod_{\ell=0}^{\Lambda} (\lambda_\ell^{0} + M^2)^{\nu_\ell/2}}$$

Mismatch in number of modes

$$\prod_{\ell=0}^{1} (\lambda_\ell^0 + M^2)^{\nu_\ell/2} = (\lambda_0^0 + M^2)^{1/2} (\lambda_1^0 + M^2)^{5/2} \approx M^6$$

Compensates M^6 factor from zero modes.

Conclusion

First, when extracting the UV cutoff dependence, it is only wise to examine any mismatches in modes between different configurations.

Second, caution should be exercised when evaluating determinants around coordinate-dependent profiles, as they may not always reduce to computations around constant backgrounds.

Also we'd like to acknowledge the initiative of Tony Gherghetta in putting the problem.