Modeling gravitational wave emission in the post-inflantionary Universe

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dedicated to memory of Valery Rubakov

Introduction

- Relic gravitational waves can be measured by the next generation of gravitational wave detectors
- The type of spectrum depends on the initial conditions and how the structures are formed
- We model the nonlinear evolution of the inflaton field and the formation of structures
- With this information, it is possible to calculate the energy-momentum tensor and, consequently, the spectrum of gravitational waves

Model

- We start with an action for an inflanton field:

$$S = \int \sqrt{-g}(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi_j - \frac{1}{2}m^2\varphi^2)d^4x \tag{1}$$

- We will use the following ansatz:

$$\varphi = \frac{e^{-imt}\psi(\vec{x},t)}{\sqrt{2}a^{\frac{3}{2}}(t)} + h.c \tag{2}$$

- Considering that the gravitational potential Φ is small and and the field changes slowly we get the action in terms of ψ:

$$S = \int d^4x\sqrt{-g}(-\frac{1}{2a^2}\partial_i\psi\partial_i\psi^* - m^2\Phi\psi\psi^* + \frac{9}{8}H^2\psi\psi^* - \frac{im}{2}(\psi\partial_0\psi^* - \psi^*\partial_0\psi) \tag{3}$$

Stress–energy tensor

- In the leading order the 0-0 stress–energy tensor component is:

$$T_0^0 = \frac{1}{a^3} m^2 \psi \psi^* \tag{4}$$

- Thus, the first Einstein equation is:

$$\Delta \Phi - 3\dot{a}^2 \Phi = \frac{4\pi G}{a} \left(|\psi|^2 - |\psi_0|^2 \right) \tag{5}$$

- And equation of motion for action (3) is:

$$i\partial_0 \psi = -\frac{\Delta \psi}{2ma^2} + m\Phi\psi - \frac{9}{8} H^2 \psi \tag{6}$$

Modes of evolution

- We have three modes of evolution:
  1. Beyond the horizon: $\dfrac{k}{a} \ll H$
  2. Linear evolution under the horizon: $\dfrac{k}{a} \gg H, m \gg H$, $\psi \approx \psi_0$
  3. Nonlinear evolution under the horizon, where $\psi \gg \psi_0$ and the formation of gravitationally bound structures takes place
- We will use these approximations in order to set the initial equations beyond the horizon, to evolve the field under the horizon in a linear mode and only then solve nonlinear equations

Linearized equations

- Let's move on to the dimensionless equations:

$$\begin{cases} x = x_0 \cdot x_{pr} \\ t = t_0 \cdot t_{pr} \\ \psi = \psi_0 \cdot \psi_{pr} \\ \Phi = \Phi_0 \cdot \Phi_{pr} \\ a = a_0 \left(\dfrac{t_{pr}}{t_*} \right)^{2/3} \end{cases} \tag{7}$$

- Then the equations will take the form:

$$\begin{cases} \Delta_{pr} \Phi_{pr} - 3\dot{a}_{pr}^2 \Phi_{pr} = \dfrac{4\pi G}{a_{pr}} \left(|\psi_{pr}|^2 - 1 \right) \\ i\partial_0 \psi_{pr} = -\dfrac{\Delta_{pr}\psi_{pr}}{2a_{pr}^2} + \Phi_{pr}\psi_{pr} - \dfrac{9}{8} H^2 \psi_{pr} \end{cases} \tag{8}$$

Linearized equations

- In (7) we have 6 parameters, but also from dimensionalization we can get connections to them, leaving only the physical ones: ψ_0 – the average value of the field and m – mass of the inflanton field.
- Then we linearize the system (8): $\psi = 1 + \delta\psi(\vec{x}, t)$ and go to the Fourier space

$$\begin{cases} -k^2\Phi - 3\dot{a}^2\Phi = \dfrac{4\pi G}{a}(\delta\psi + \delta\psi^*) \\ i\partial_0\delta\psi = \dfrac{k^2\delta\psi}{2a^2} + \Phi - \dfrac{9}{8}H^2(1+\delta\psi) \end{cases} \tag{9}$$

- These equations are true both beyond and below the horizon, but only at the linear stage of evolution. They are easy to solve numerically and we can use them to set initial conditions beyond the horizon and evolve the field to reach the nonlinear stage below the horizon

Nonlinear equations under the horizon

- Under the horizon we can use the approximation $H \ll m$. Than (8) will be:

$$\begin{cases} i\dfrac{d\psi_{pr}}{dt_{pr}} = -\dfrac{\Delta_{pr}\psi_{pr}}{2a_{pr}^2} + \Phi_{pr}\psi_{pr} \\ \Delta_{pr}\Phi_{pr} = \dfrac{4\pi}{a_{pr}}(|\psi_{pr}|^2 - 1) \end{cases} \tag{10}$$

- To solve the second equation, we will use the Fourier series expansion method:

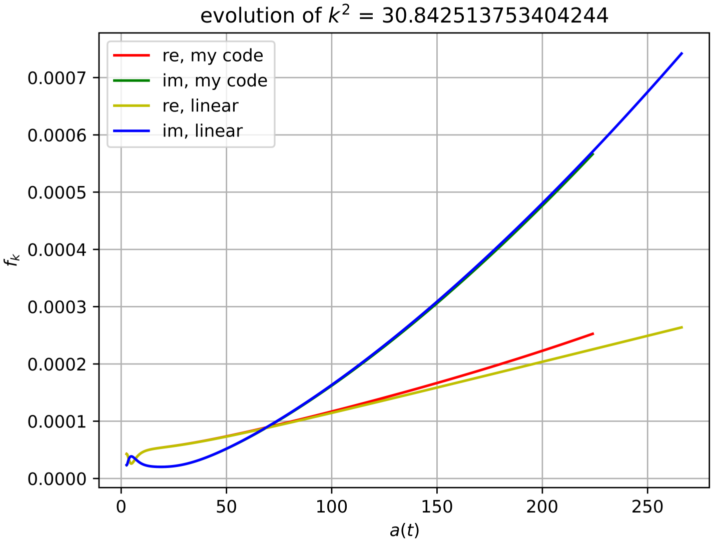
$$\tilde{\Phi}(\vec{k}) = -4\pi G \frac{\tilde{\varrho}(\vec{k})}{k^2} \tag{11}$$

- To solve the first equation, we will use the symplectic method of the 4th order:

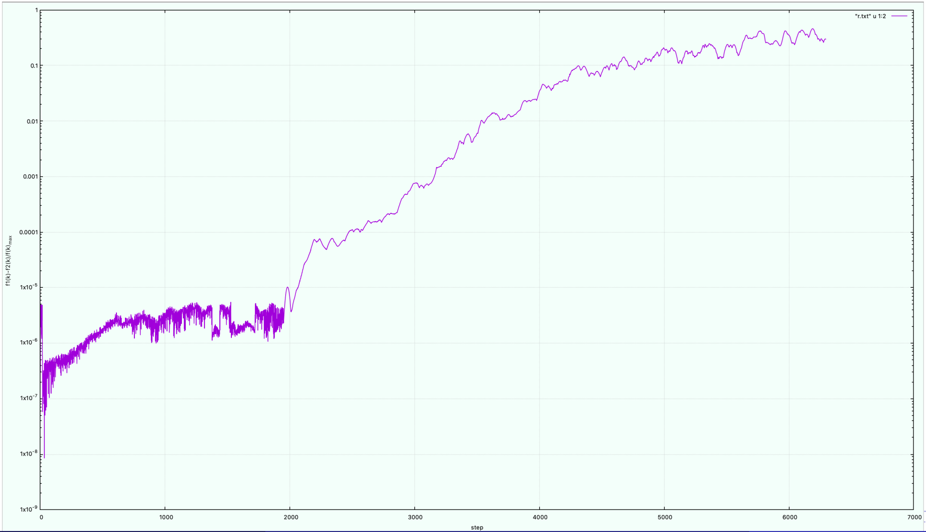
$$\mathbf{f}(t) = e^{-it\mathbf{H}} f_0 \tag{12}$$

$$e^{\tau\mathbf{H}} = e^{a_n\tau\mathbf{V}} \cdot e^{b_n\tau\mathbf{T}} + o(\tau^4), \tag{13}$$

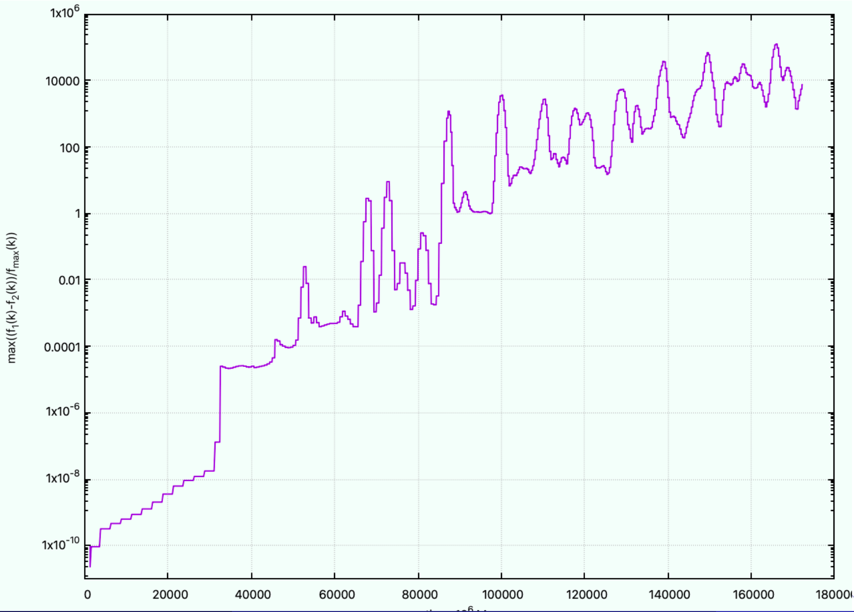
Nonlinear and linear solutions



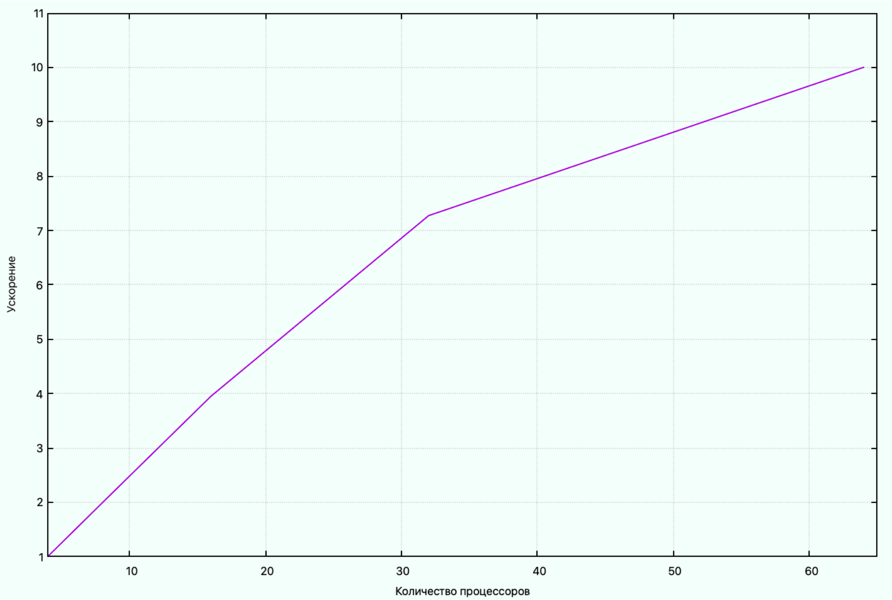
The difference of solutions on lattices 128/256 in Fourier space



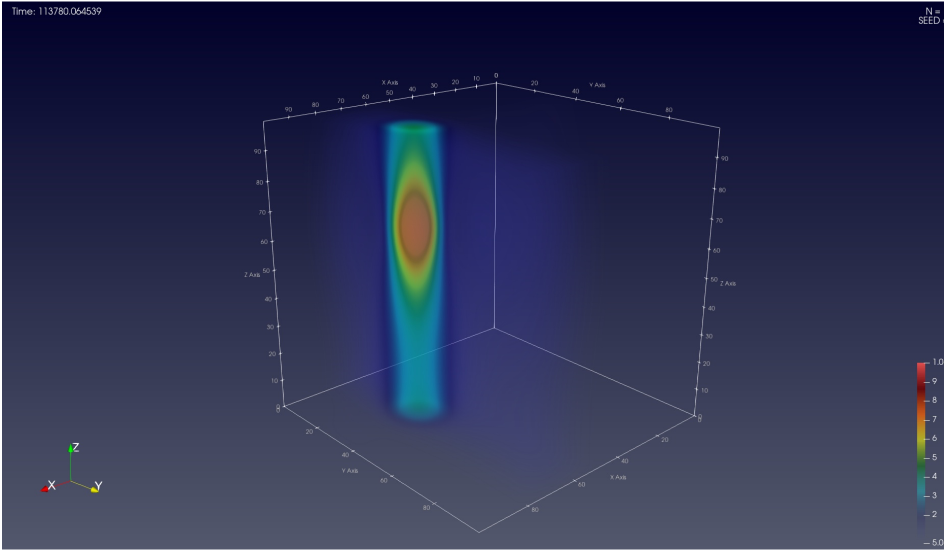
The mean value in Fourier space at large k



MPI acceleration: acceleration(number of processors)



Evolution of the field. Color is $|\psi|^2$



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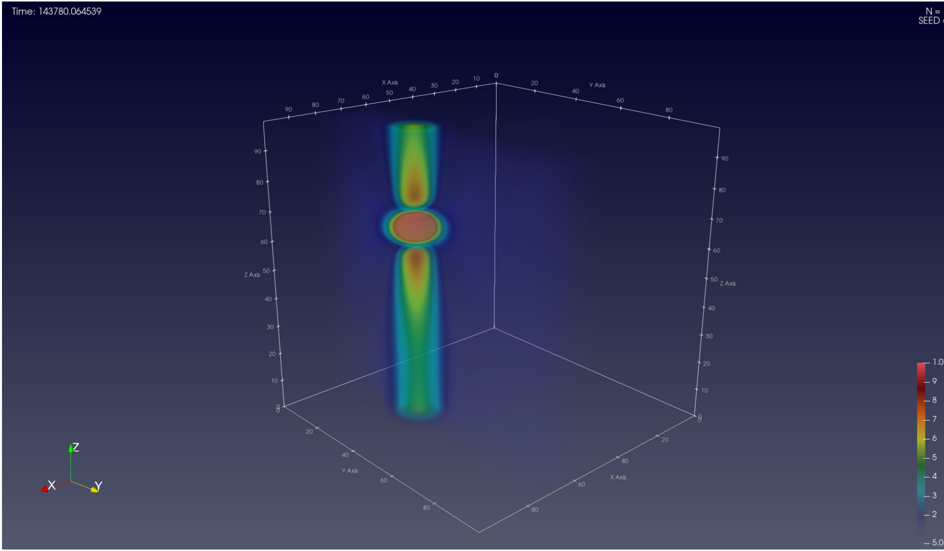

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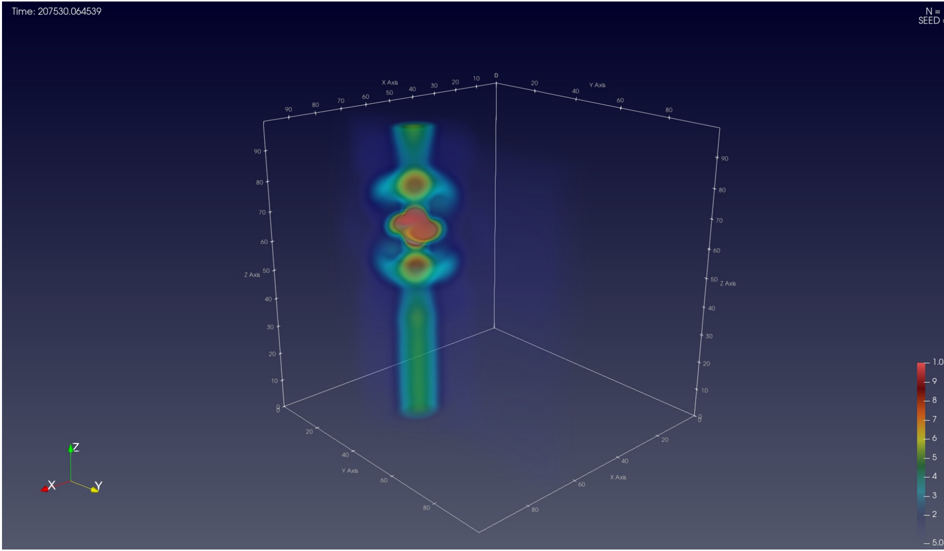

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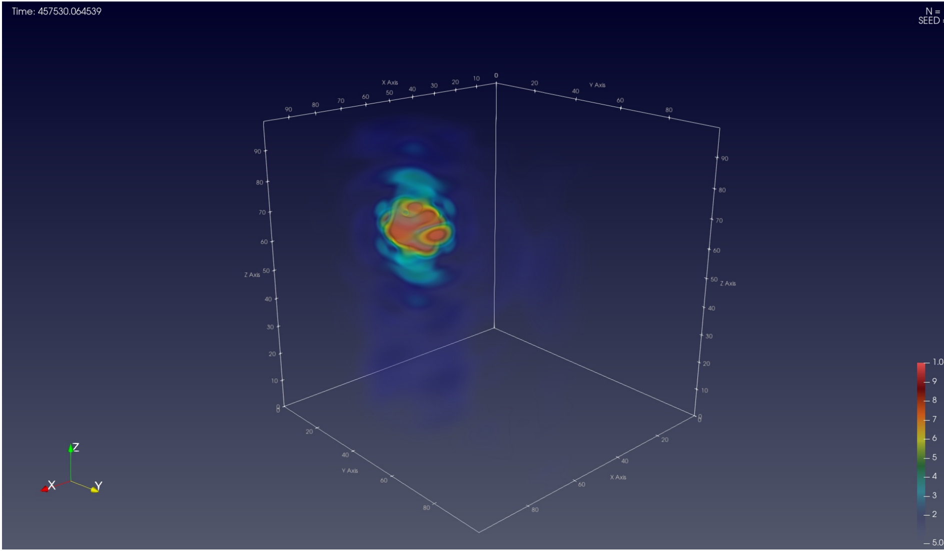

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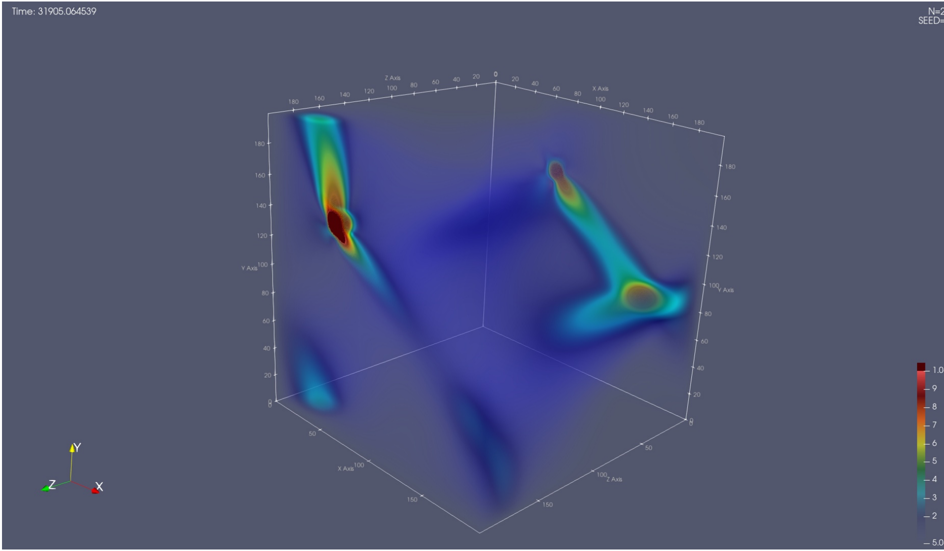
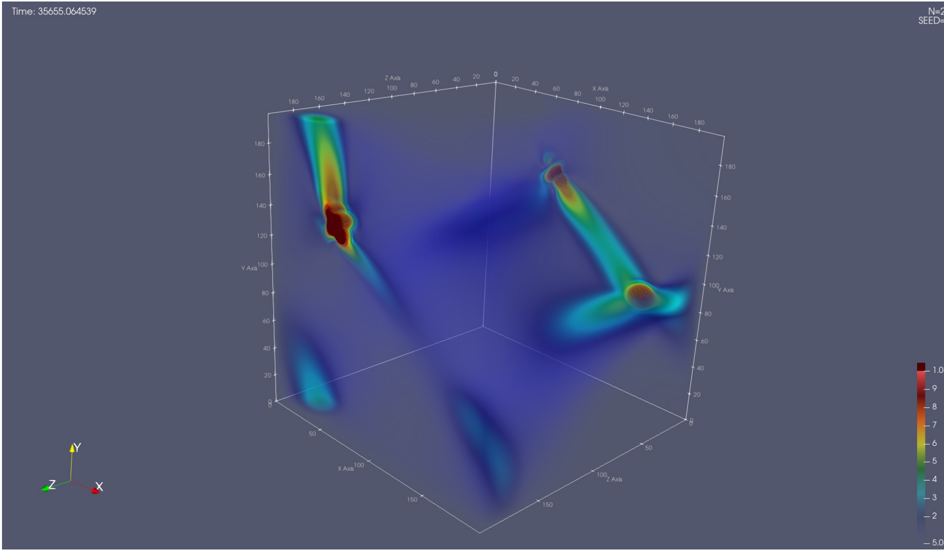
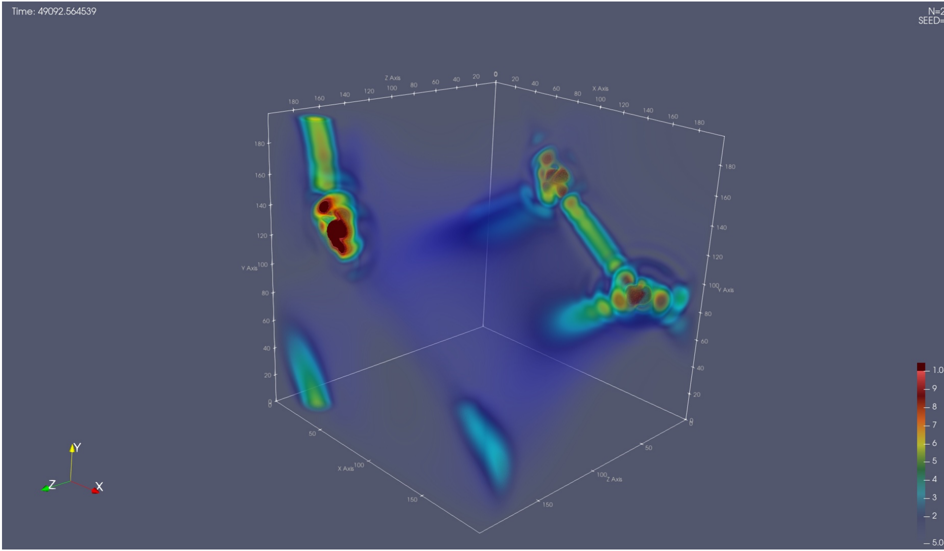


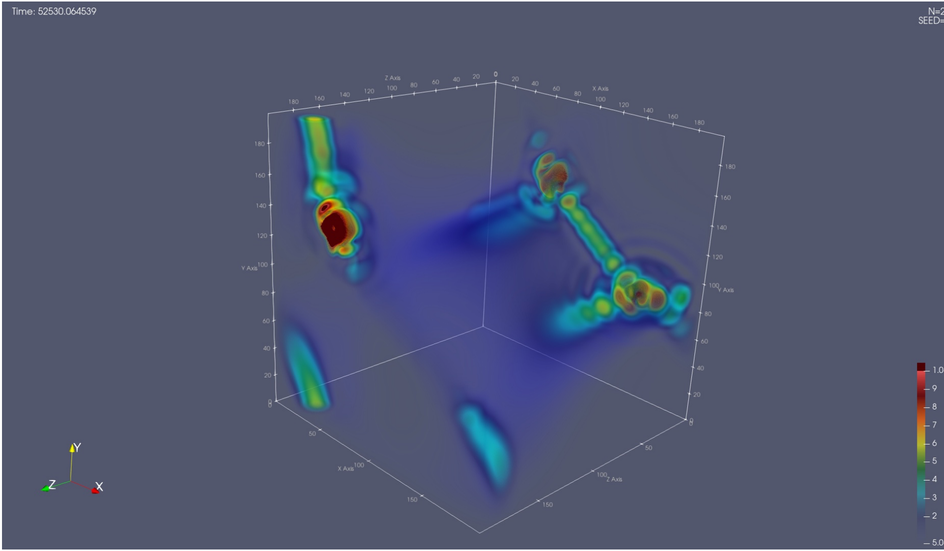
Evolution of the field. Color is $|\psi|^2$



Evolution of the field. Color is $|\psi|^2$



Evolution of the field. Color is $|\psi|^2$



- The equation for tensor modes in Fourier space has the form

$$\ddot{h}_{ij}(\vec{k},t) + 3H\dot{h}_{ij}(\vec{k},t) - \frac{\nabla^2}{a^2}h_{ij}(\vec{k},t) = 16\pi G\Pi_{ij}^{TT}(\vec{k},t) \tag{14}$$

- Where $\Pi_{ij} = \frac{1}{a^2}\left(T_{ij} + pa^2(\delta_{ij} + h_{ij})\right)$ We will make a replacement:

$$H_{ij}(\vec{k},\eta) = ah_{ij}(\vec{k},\eta) \tag{15}$$

- Then the equation will be written as, where $' \equiv \frac{d}{d\eta}$:

$$H_{ij}''(\vec{k},\eta) + \left(k^2 - \frac{a''}{a}\right)H_{ij} = 16\pi Ga\Pi_{ij}^{TT}(\vec{k},\eta) \tag{16}$$

- Let's neglect the second term in parentheses ($k^2 \gg a''/a$):

$$H_{ij}''(\vec{k},\eta) + k^2H_{ij} = 16\pi Ga\Pi_{ij}(\vec{k},\eta) \tag{17}$$

- We believe that after a certain moment $\eta = \eta_f$ no gravitational waves were emitted, so for $\eta > \eta_f$ the solution can be written as follows:

$$H_{ij}(\eta, \vec{k}) = A_{ij}(\vec{k})e^{ik(\eta-\eta_f)} + A^*_{ij}(\vec{k})e^{-ik(\eta-\eta_f)} \tag{18}$$

where

$$\begin{cases} A_{ij}(\vec{k}) = \frac{8i\pi G}{k} \int_{\eta_i}^{\eta_f} d\eta' e^{ik(\eta_f-\eta')} a(\eta')\Pi^{TT}_{ij}(\eta', \vec{k}) \\ A^*_{ij}(\vec{k}) = -\frac{8i\pi G}{k} \int_{\eta_i}^{\eta_f} d\eta' e^{-ik(\eta_f-\eta')} a(\eta')\Pi^{TT}_{ij}(\eta', \vec{k}) \end{cases} \tag{19}$$

and $\Pi_{ij} = \dfrac{1}{2a^5}(\partial_i\psi^*\partial_j\psi + \partial_i\psi\partial_j\psi^*)$

- Energy density of gravitational waves:

$$\rho_{gw} = \frac{1}{32\pi G a^4}\langle H'_{ij}(\eta, \vec{x})H'_{ij}(\eta, \vec{x})\rangle = \frac{1}{32\pi G a^4}\frac{1}{V}\int d^3\mathbf{k} H'_{ij}(\eta, \vec{k})H'^{*}_{ij}(\eta, \vec{k}) \tag{20}$$

$$\rho_{gw} = \frac{8\pi G}{a^4}\frac{1}{V}\int d^3\mathbf{k} \left| \int_{\eta_i}^{\eta_f} d\eta' e^{ik(\eta_f-\eta')} a(\eta')\Pi^{TT}_{ij}(\eta', \vec{k}) \right|^2 \tag{21}$$

The spectrum

- We define the energy density spectrum of gravitational waves per unit logarithmic interval as follows:

$$\left(\frac{d\rho_{gw}}{d\ln k}\right)_{\eta>\eta_f} = \frac{S_k(\eta_f)}{a^4(\eta)} \tag{22}$$

- Then we can write it as follows, where we took into account that gravitational waves have two polarizations and summation follows them

$$S_k(\eta) = \frac{8\pi G k^3}{V}\int d\Omega \sum_{p=+,\times}\left[\left|\int_{\eta_i}^{\eta_f} d\eta' e^{ik(\eta_f-\eta')}a(\eta')\Pi_p(\eta',\vec{k})\right|^2\right. \tag{23}$$

Conclusions

- We can simulate the nonlinear evolution of the field and the formation of structures
- The initial conditions are set beyond the horizon. We can set different types of primordial fluctuations
- In the future, it is planned to obtain a spectrum of gravitational waves that are emitted due to the nonlinear evolution of the field

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