Suppression exponent for multiparticle production in $\lambda\phi^4$ theory

based on 2212.03268 and 2111.04760

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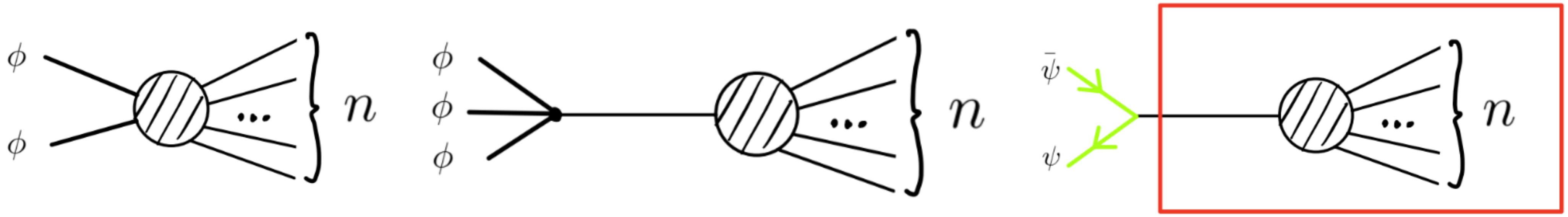
Multiparticle production

- Multiparticle probabilities
- Perturbation theory
- Exponentiation

Multiparticle processes

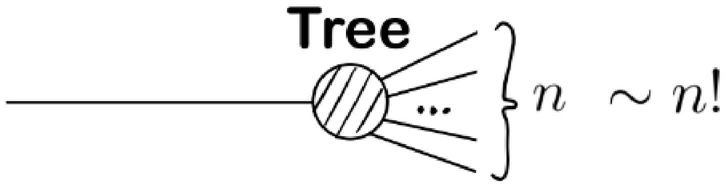
$n \gg 1$ bosons in the final state and $n_i \ll \lambda^{-1}$ in the initial

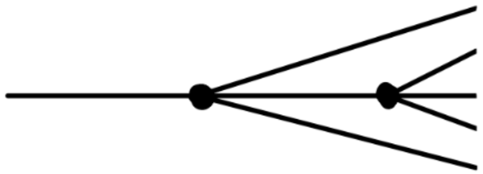
Examples



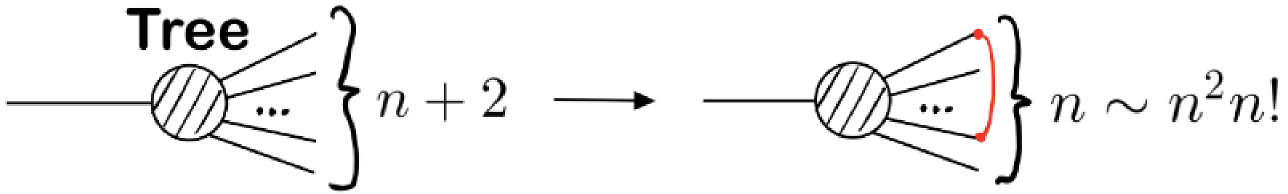
Counting diagrams

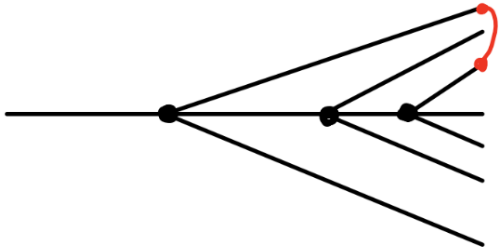
The number of tree diagrams $\propto n!$





The number of one-loop diagrams $\propto n!\, n^2$





Amplitudes on threshold

In theory $S[\phi] = \frac{1}{2}\int d^4x \left(-\phi\Box\phi - m^2\phi^2 - \frac{\lambda\phi^4}{2}\right)$

$$A_{1\to n} = \langle n, E = nm|\,\hat{\phi}\,(0)\,|0\rangle$$

ϕ_{cl} is a generating functional

$$\langle n|\,\phi\,(x)\,|0\rangle = \prod_{a=1}^{n}\int (d^4x_a) e^{ip_a x_a}\left(-p_a^2 + m^2\right)\frac{\delta}{\delta\rho\,(x_a)}\,\langle 0+|\,\phi\,(x)\,|0-\rangle^{\rho}\,|_{\rho=0}$$

$$S \longrightarrow S + \int d^4x\rho\phi \qquad \langle 0+|\,\phi\,(x)\,|0-\rangle^{\rho} \equiv \phi_{\text{cl}}\,(x)$$

Tree-level and one loop

ϕ_{cl}^{tree} — solution of the classical equation with $E = 0$ [Brown, 1992]

$$A_{1\to n}^{tree} = n!\left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}, \text{n — odd}$$

Indeed factorial growth!

One-loop correction with ϕ_{cl}^{tree} as a background [Voloshin, 1992]

$$A_{1\to n}^{1-loop} = A_{1\to n}^{tree}\, B\lambda\left(n^2 + O(n)\right)$$
$$B = const, n \gg 1$$

Indeed has $n!\, n^2$

Amplitudes and probabilities

Correction near the threshold [Libanov et al., 1994]

$$A^{tree}_{1\to n}(p_1, \dots, p_n) \approx A^{tree}_{1\to n}(0)e^{-\frac{5}{6}\varepsilon n}, \qquad \varepsilon = \frac{E}{n} - m \ll m$$

Probability estimation

$$P_{1\to n}(\varepsilon) = \int \frac{|A_{1\to n}|^2}{n!} \prod_i \frac{d^3 p_i}{2\omega_i} \approx \frac{|A_{1\to n}(0)|^2}{n!} \times (\text{phase vol.}) \propto n!$$

Growth persists!

Failure of perturbation theory

Perturbative series for $A_{1\to \mathrm{n}}$ [Argyres, 1993]

$$A_{1\to n} = n!\left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}\left[1 + \#\lambda(n^2 + \cdots) + \#\lambda^2(n^4 + \cdots) + \cdots\ \right]$$

At L loops leading at $n \gg 1$ contribution $\propto n!\,(\lambda n^2)^L$

Blow up at $n \gtrsim \lambda^{-1}$

Series resummation

Resummation of leading at $n \gg 1$ contributions in all loops gives

$$A_{1\to n}^{resummed} = A_{1\to n}^{tree} \exp(B\lambda^2 n^2/\lambda)(\dots)$$

B is the same as in the one-loop $\propto n^2$ term[Libanov et al., 1994]

At $n \gg 1$

$$A_{1\to n}^{tree} \sim \sqrt{n!}\exp(\lambda n/2\lambda \ln \lambda n - \lambda n/2\lambda)$$

Amplitude

$$A_{1\to n}^{resummed} \sim \sqrt{n!}\exp(F_A(\lambda n)/\lambda)$$

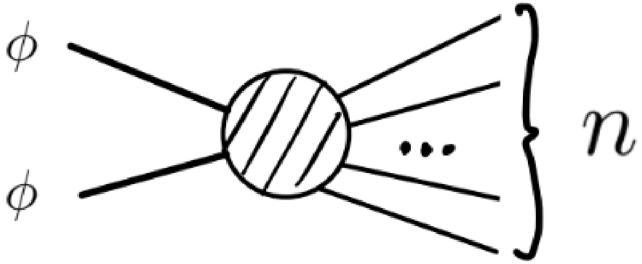
Exponential form $\stackrel{?}{=}$ Semiclassical treatment

The exponent

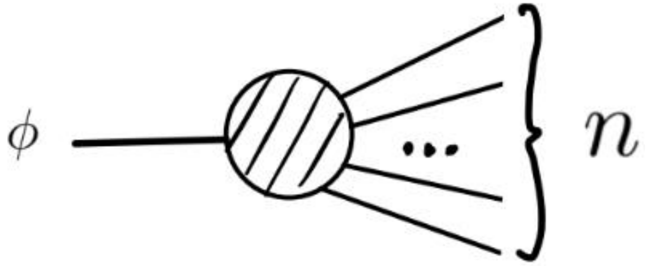
Conjectures [Libanov et al., 1994]

1. $P_{few\to n}(\varepsilon) \sim e^{F(\lambda n,\varepsilon)/\lambda}$, $n \gg 1, \lambda n, \varepsilon = const.$ $F(\lambda n,\varepsilon)$— “holy grail” function
2. $F(\lambda n,\varepsilon)$ does not depend on the initial state if $n_i \ll \lambda^{-1}$

Semiclassical limit: $\lambda \to 0; \lambda n, \varepsilon = const$

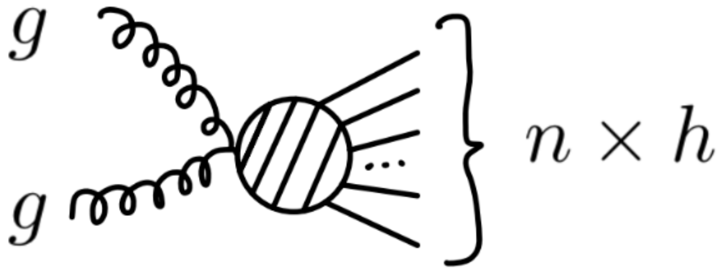


or



etc.

"Higgsplosion"



It was suggested [Khoze et al., 2017] that at high energies

probability $\propto e^{F_{Higgsplosion}/\lambda}$ grows with n

$$F_{Higgsplosion} = \lambda n \ln\left(\frac{\lambda n}{4}\right) + \frac{3}{2}\lambda n \ln\frac{\varepsilon}{3\pi m} + \frac{\lambda n}{2} + 0.845(\lambda n)^{3/2}, \quad n \leq n_*, E = n(m+\varepsilon)$$

$F_{Higgsplosion} = 0$ at n_* and still grows for $n > n_*$ (unitarity?)

Result was obtained semiclassically with additional assumptions

Consistency check?

Method of singular solutions

- Formulation
- Numerical implementation
- Verification

[Son, 1995]

Landau method in QM

<u>In QM one can consider</u>

$$\langle E' | \hat{O} | E \rangle \sim e^{f}, \quad f = -\mathrm{Im}\left[\int^{x_*} [2m(E'-V)]^{1/2}\, dx - \int^{x_*} [2m(E-V)]^{1/2}\, dx \right]$$

$$x_0 \; \bullet\!\!\longrightarrow\!\!\bullet \; x_*$$

$$E > V(x_0) \qquad\qquad V(x_*) = \infty$$

- $\hat{O}$ can be $\hat{x}, \hat{x}^2, \ \hat{p}$, etc. — answer is insensible
- x_* is a singular point of $V \Rightarrow$ singular solutions in path integral
- We need only exponential accuracy

Probability

Our aim is inclusive probability

$$P_n(E) \equiv \sum_f \left| \langle f; E, n | \hat{S}\hat{O} | 0 \rangle \right|^2 = \int D[f,\phi] e^{W/\lambda} \sim \mathrm{e}^{F(\lambda n, \varepsilon)/\lambda},$$

where

$$\varepsilon = E/n - m$$

with $n_i \ll \lambda^{-1}$ particles

$\hat{O}|0\rangle$ — initial state

We also use $\phi \to \phi/\sqrt{\lambda}$ to extract λ^{-1} from action

Initial state

$$\hat{O}_J|0\rangle = \exp\left(-\frac{1}{\lambda}\int d^3\mathbf{x}\, J(\mathbf{x})\hat{\phi}(0,\mathbf{x})\right)|0\rangle$$

1. Creates $\mathrm{n_J} \propto J^2/\lambda$ particles
2. $n_J \ll \lambda^{-1}$ or $J \to 0 \Rightarrow$ universality
3. At $1 \ll n_J \ll \lambda^{-1}$ we can
   - Calculate semiclassically
   - Use universality

Path integral representation

We compute

$$P_n^J(E) \equiv \sum_f \left|\langle f; E, n|\, \hat{S}\hat{O}_J |0\rangle\right|^2 = \int D[f,\phi]\, e^{W_J/\lambda},$$

Universality: $P_n(E) = \lim_{J\to 0} P_n^J(E)$

In the path integral representation

$$P_n^J(E) = \int \mathbf{D}f\, |A_J|^2 \qquad A_J = \int \mathbf{D}\phi_{i,f} \langle f; E, n|\, \phi_f\rangle \langle\phi_f|\, \hat{S}\hat{O}_J\, |\phi_i\rangle\, \langle\phi_i|\, 0\rangle$$

$e^{B_f(\phi_f,\, E,\, n)}$ $\quad e^{iS[\phi] - \int J\phi}$ $\quad e^{B_i(\phi_i)}$

Saddle point

Use saddle-point approximation at $\lambda \to 0$

$$\frac{\delta W_J}{\delta \phi_{\text{cl}}} = 0$$

$$P_n^J(E) \sim e^{F_J(\lambda n,\, \varepsilon)/\lambda}$$

$$F_J(\lambda n, \varepsilon) = W_J[\phi_{\text{cl}}]$$

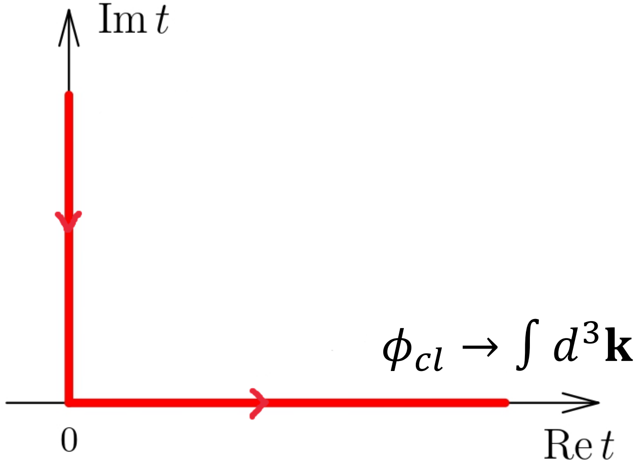
ϕ_{cl} obeys classical field equation

$$\Box \phi_{cl}(x) + m^2 \phi_{cl}(x) + \phi_{cl}^3(x) = iJ(\mathbf{x})\delta(t)$$

Boundary conditions

Solution is calculated on the complex time contour

$\phi_{cl} \to 0$ as $t \to +i\infty$
(vacuum)



$\phi_{cl} \to \int d^3\mathbf{k}\left(a_{\mathbf{k}} \mathrm{e}^{-i\omega_{\mathbf{k}}t} + b^*_{\mathbf{k}} \mathrm{e}^{i\omega_{\mathbf{k}}t}\right)$ as $t \to +\infty$

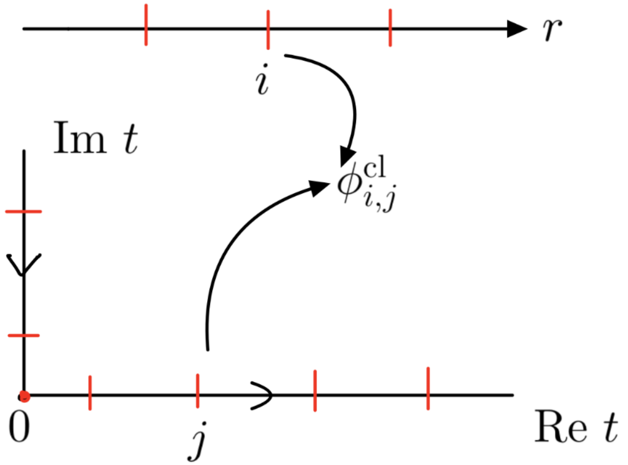
$$a_{\mathbf{k}} = e^{-\theta+2T\omega_{\mathbf{k}}} b_{\mathbf{k}}$$

T, θ — Lagrange multipliers (E, n fixing)

Numerical implementation

To solve the saddle-point boundary value problem numerically we

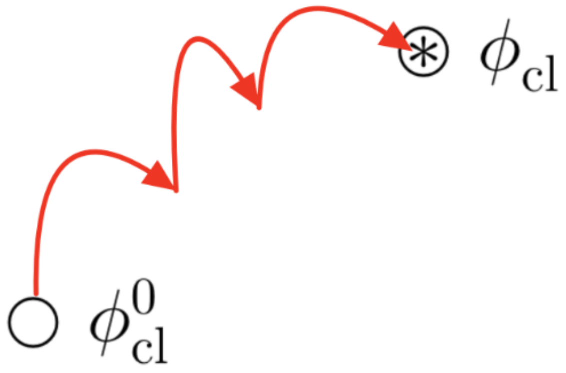
- Use $J(\mathbf{x}) = j_0 e^{-\mathbf{x}^2/2\sigma^2}$
- Discretize:



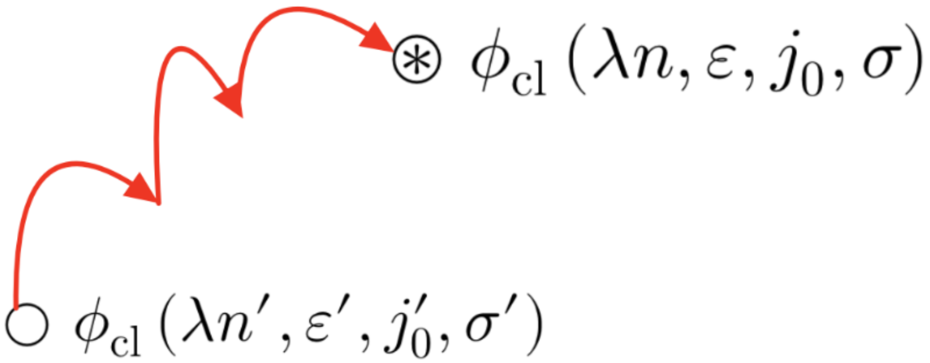
- Solve $2 \times N_r \times N_t + 2$ real non-linear equations
- We consider only spherically symmetrical $\phi_{\rm cl}$

Solving the equations

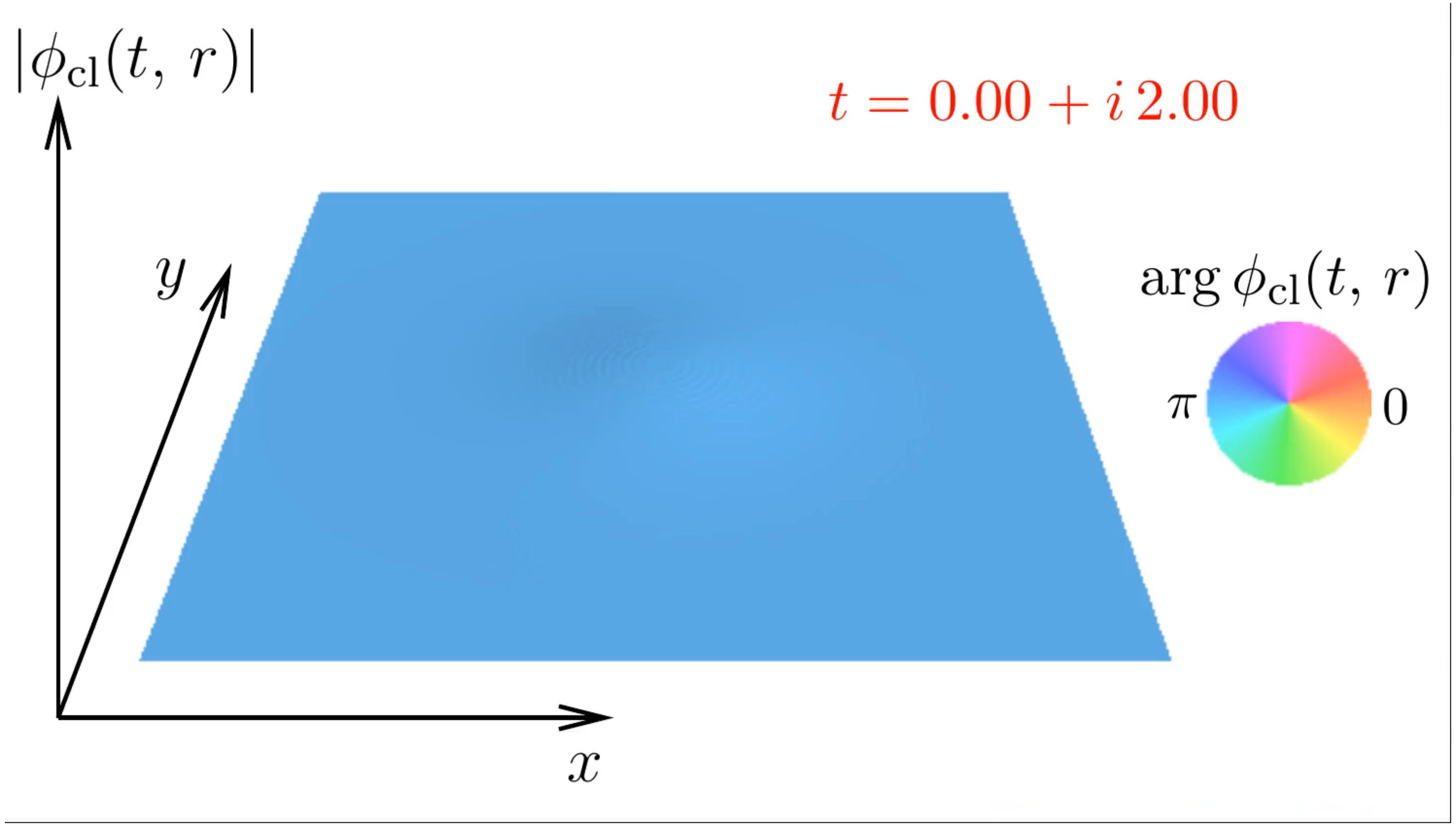
- We used Newton-Raphson method



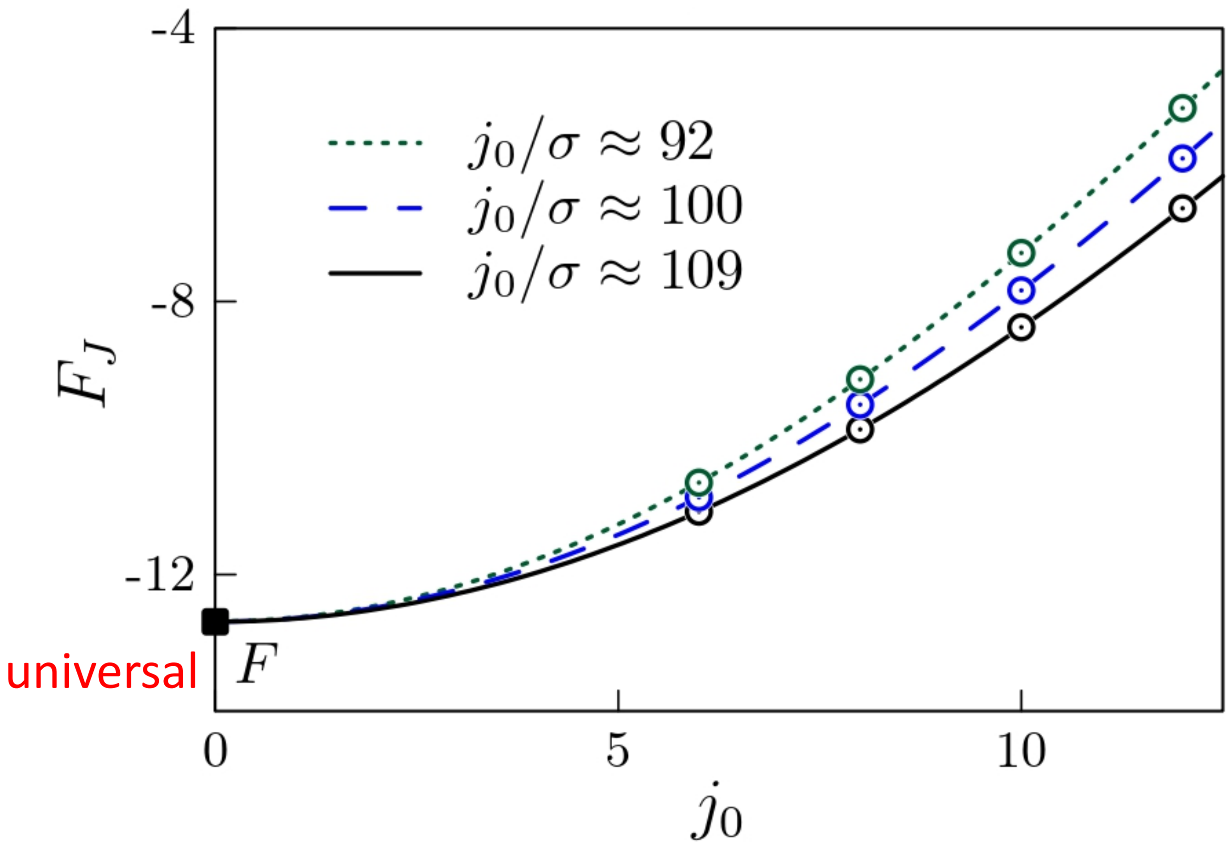
- Solution with required parameters is obtained by walking in the parameter space



Example of a solution



Extrapolation $J \to 0$

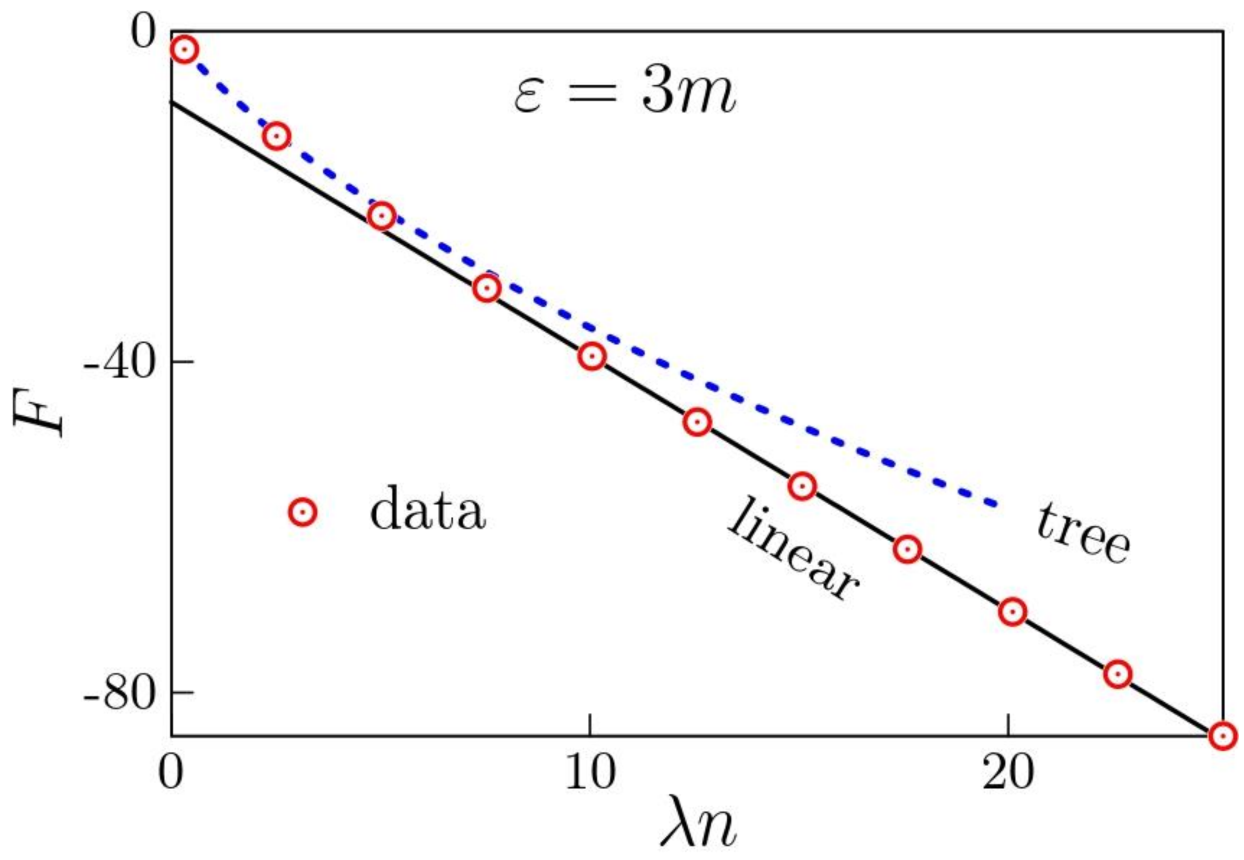


- $J(\mathbf{x}) = j_0 e^{-\mathbf{x}^2/2\sigma^2}$
- Consider $j_0 \to 0$, $j_0/\sigma = const$ — weak narrow source
- Solutions become singular at $j_0 = 0$ because of BC
- Use $F_J = F + F_2 j_0^2 + F_4 j_0^4 + \cdots$
- Can compute $j_0 \to 0$ with different j_0/σ

Probabilities as functions of $\lambda n, \varepsilon$

- Suppression exponent and amplitudes
- Limit $\lambda n \gg 1$
- Limit $\varepsilon \gg m$

Example of typical behavior



- Tree-level (known) [Bezrukov, 1998]:

$$F = \lambda n \ln\left(\frac{\lambda n}{16}\right) - \lambda n + \lambda n\, f(\varepsilon) + O(\lambda n)^2$$

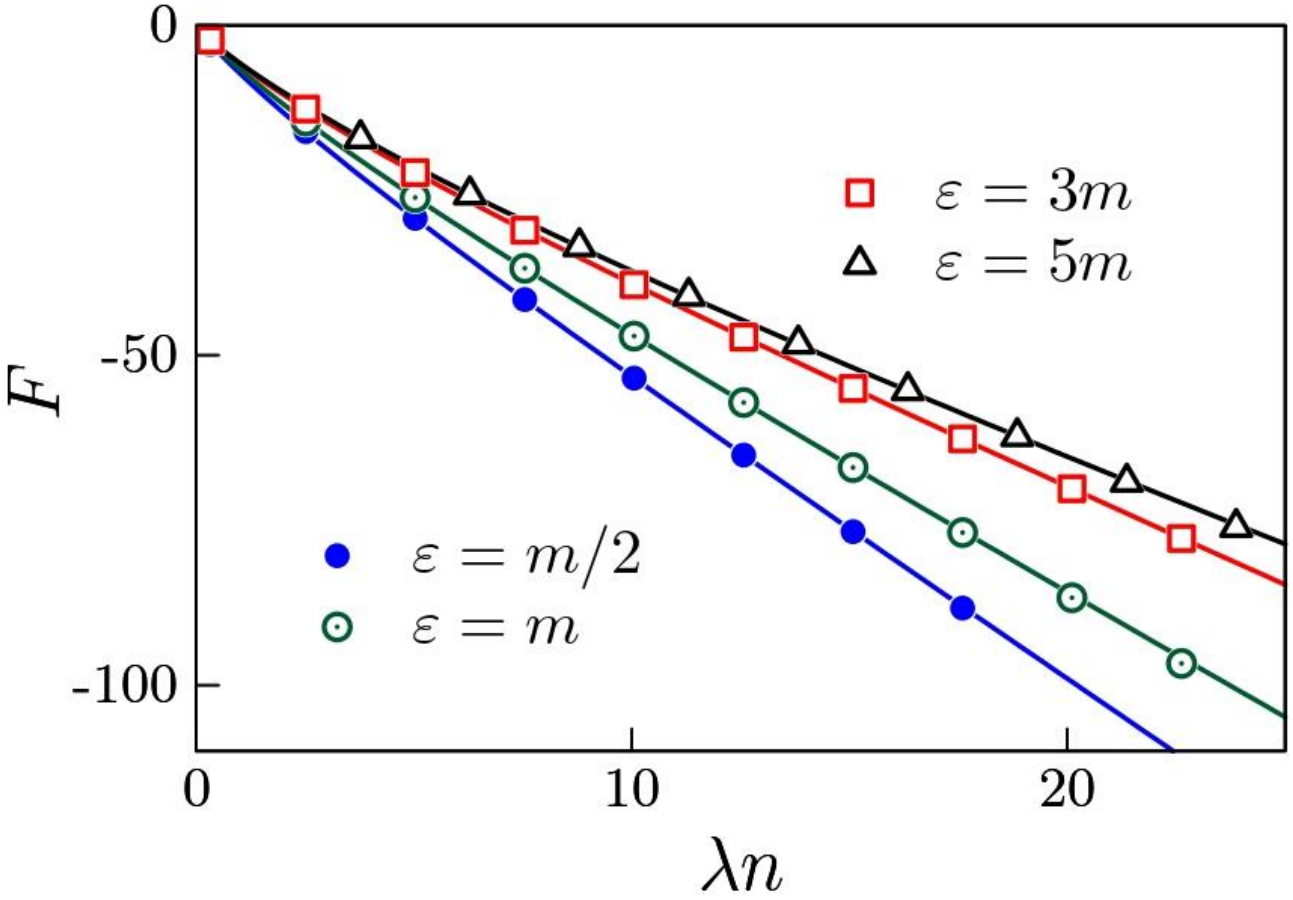
- Linear with good precision:

$$F = f_\infty(\varepsilon)\lambda n + g_\infty(\varepsilon)$$

$$P_n(E) \sim e^{f_\infty(\varepsilon)n + g_\infty(\varepsilon)/\lambda}$$

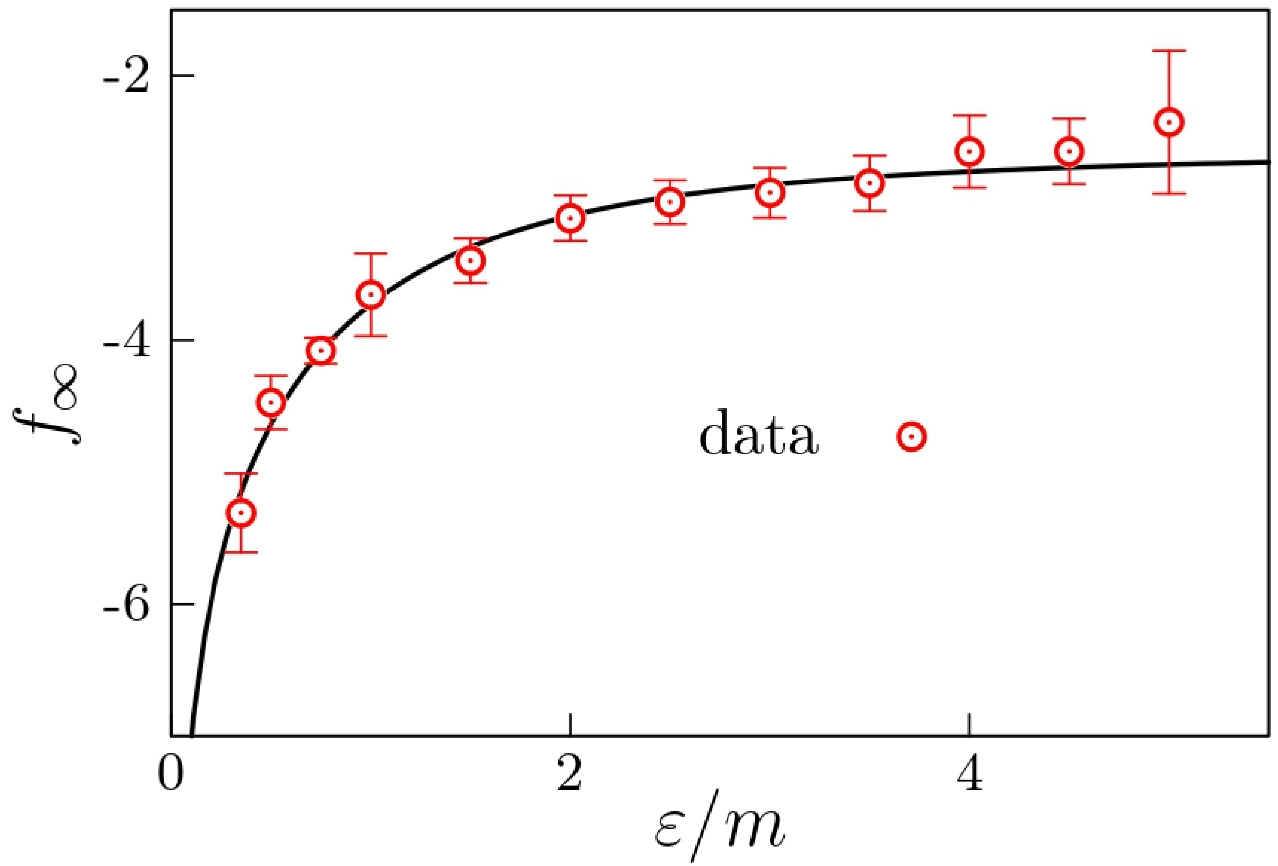
- $\varepsilon = \frac{E}{n} - m$
- $P_n(E) \sim e^{F/\lambda}$

Numerical suppression exponents



- $\varepsilon = \frac{E}{n} - m$
- F curves become closer to each other, when ε grows

f_∞ as a function of ε



- $F \to f_\infty(\varepsilon)\lambda n + g_\infty(\varepsilon)$ for $\lambda n \gg 1$
- f_∞ grows to -2.57 ± 0.06 for $\varepsilon \to \infty$
- $\varepsilon = \frac{E}{n} - m$

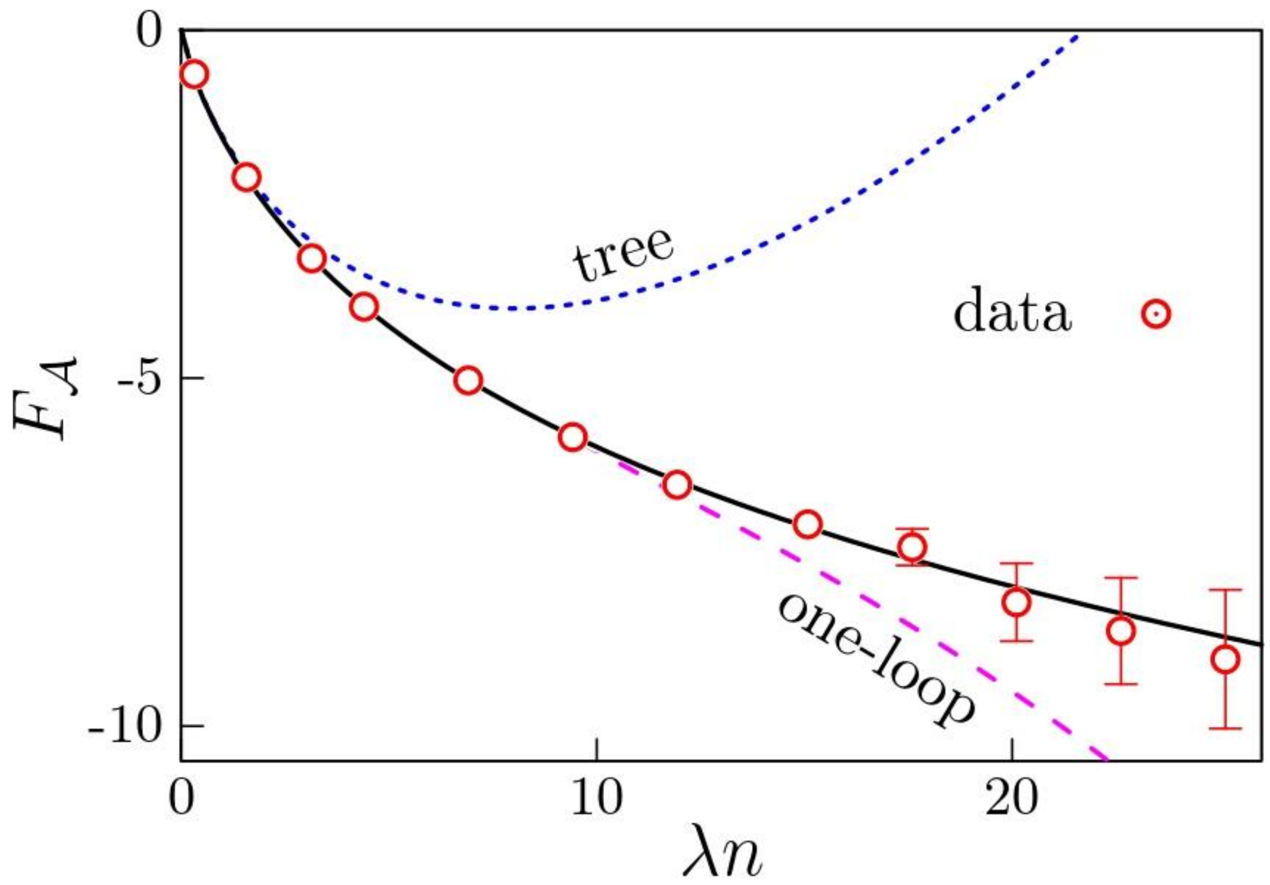
Amplitudes at threshold ($\varepsilon \to 0$)

In the limit $\varepsilon \to 0$ one can estimate

$$P_n(E \to mn) \approx |A_n|^2 \times \text{phase volume} \approx e^{F(\lambda n, \varepsilon \leq m)}$$

Can get $|A_n| = \exp(F_A/\lambda)$ from $F(\lambda n, \varepsilon \leq m)$ via extrapolation $\varepsilon \to 0$

Fitting of $F_A(\lambda n)$



- $F_A = \frac{\lambda}{2} \lim_{E \to nm} \ln \frac{P_n(E) m^{4-2n}}{\text{phase vol.}}$
- Tree-level:

$$F_A^{\text{tree}} = \frac{\lambda n}{2} \ln\left(\frac{\lambda n}{8}\right) - \frac{\lambda n}{2}$$

- One-loop:

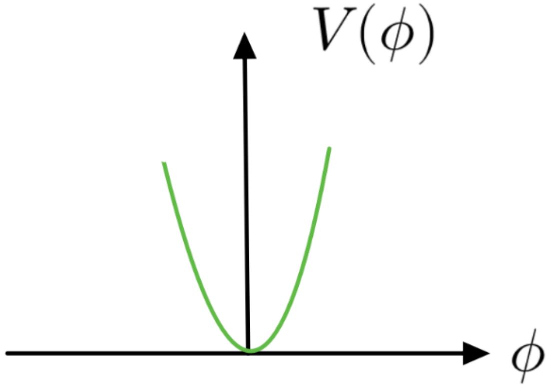
$$F_A^{1-\text{loop}} = F_A^{\text{tree}} + \frac{(\lambda n)^2 3^{3/2}}{32\pi^2} \ln(2 + \sqrt{3})$$

Conclusions

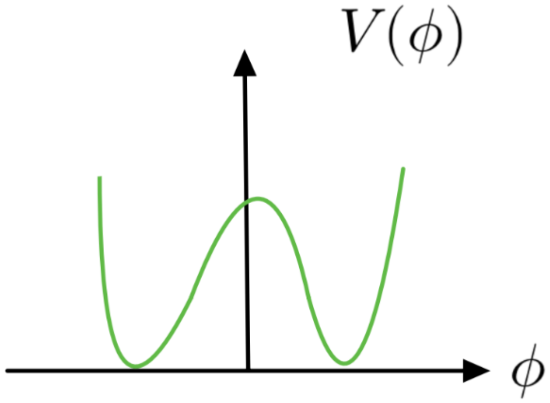
We calculated "holy grail" function $F(\lambda n, \varepsilon)$

$$P_n(E) \xrightarrow[\lambda n \to +\infty]{} e^{n f_\infty + g_\infty / \lambda}$$

Generic?



Suppression



?

Backup slides

Limit $J \to 0$ in more detail

After ϕ_{cl} is found, we calculate

$$F_J = 2\lambda ET - \lambda n\theta - 2\lambda \mathrm{Im} S[\phi_{cl}] - 2\mathrm{Re}\int d^3\mathbf{x}\, J(\mathbf{x})\phi_{cl}(0,\mathbf{x})$$

Then take the limit

$$F(\lambda n, \varepsilon) = \lim_{J\to 0} F_J(\lambda n, \varepsilon)$$

Solutions become singular in the limit

- $E_i = 0,\ E_f = E \Rightarrow$ discontinuity:
- $iJ(\mathbf{x})\delta(t) \Rightarrow$ energy changes at $t = 0$
- $J = 0 \Rightarrow$ energy conservation conflicts with BC

Source-dominated ϕ_{cl}^{0}

When $\lambda n \ll 1; \sigma, \lambda E = const$ and $\lambda n \propto j_0^2$

only source produces particles

$$\Box\phi_{cl} + m^2\phi_{cl} + \cancel{\phi_{cl}^3} = i j_0 e^{-\mathbf{x}^2/2\sigma^2}$$

$$\phi_{cl} \sim j_0 \qquad m^2\phi_{cl} \sim j_0 \qquad \phi_{cl}^3 \sim j_0^3$$

Can be analytically solved!

We use the solution as ϕ_{cl}^{0}

Fits for F

Fitting function for F must

- Be close to tree-level $\lambda n \ln\left(\frac{\lambda n}{16}\right) - \lambda n + \lambda n\, f(\varepsilon)$ up to $O(\lambda n)^2$
- Tend to linear function for $\lambda n \to +\infty$

We used function with two fitting parameters:

$$F \approx \lambda n f_\infty(\varepsilon) - \frac{\lambda n}{2} \ln\left[\left(\frac{16}{\lambda n}\right)^2 e^{2-2f(\varepsilon)+2f_\infty(\varepsilon)} - \frac{2g_\infty(\varepsilon)}{\lambda n} + 1\right]$$

Fit for f_∞

Fitting function for $f_\infty(\varepsilon)$ must

- Logarithmically diverge for $\varepsilon \to 0$
- Tend to constant for $\varepsilon \to +\infty$

We used function with two fitting parameters:

$$f_\infty(\varepsilon) \approx -\frac{3}{4}\ln\left[\left(\frac{d_1 m}{\varepsilon}\right)^2 + d_2\right], \qquad d_i \approx \{10.7, 30.7\}$$