General radially moving references frames in the black hole background

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$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 (d\theta^2 + d\varphi^2 \sin^2\theta).$$

$$d\tilde{t} = e_0 dt + \frac{dr}{f} P_0$$

$$P_0 \equiv \sqrt{e_0^2 - f}.$$

If we use original radial coordinate, we have

$$d\tilde{t} = e_0 dt + \frac{P_0 dr}{f},$$

instead of T we use in this particular case $\tilde{t}$

$$ds^2 = -d\tilde{t}^2 + 2\frac{d\tilde{t}drV}{e_0} + \frac{dr^2}{e^2{}_0}$$

Generalization of GP metric

$e_0 \to 0$ Singular transformation. Both coordinates fail to be independent

$$d\tilde{t} \sim dr$$

Impossible to take limit in this metric directly

The proper distance grows indefinitely, metric becomes singular

$$d\tilde{t} = e_0 dt - \frac{P_0 dr}{f},$$

Contracting-expanding

What happens in the limit $e_0 \to 0$

Singular transformation. Term with $d\rho$ drops out

Works under horizon only, f<0 $P_0 = \sqrt{e^2{}_0 - f}$

For a synchronous metric the limit is allowed, provided we make rescaling

$$\rho = e_0 \tilde{\rho}$$

$$ds^2 = -dT^2 + g(r(T))d\tilde{\rho}^2 + r^2 d\Omega^2 \qquad g = -f > 0$$

$$T = -\int^r \frac{d\tilde{r}}{g(\tilde{r})}$$

Novikov presentation,
Particular case of Kantowski-Sachs cosmology

Another version: modification of GP system under horizon

From $\tilde{t}, r$ to $\tilde{t}, t$

$$ds^2 = -\frac{g}{P^2{}_0} d\tilde{t}^2 + \frac{g^2 dt^2}{P^2{}_0} + 2\frac{e_0 g dt d\tilde{t}}{P^2{}_0} + r^2(t,\tilde{t}) d\Omega^2$$

$$ds^2 = -d\tilde{t}^2 + \frac{g^2}{P^2{}_0}(dt + \frac{e_0 d\tilde{t}}{g})^2 + r^2(t,\tilde{t}) d\Omega^2$$

Under horizon t is spacelike, so we have 1 spacelike and 1 timelike coordinates
In GP system two timelike under horizon.
Nondiagonal term defines flow velocity $-\frac{e_0}{g}$

can be interpreted as a velocity with respect to frame where fiducial observer has $e_0 = 0$
Metric dual to GP, arranged for region under horizon, has smooth limit to
$e_0 = 0$

Velocities and their behavior

Frame attached to free falling observer

$$e_0$$

Energy of observer

$$e$$

energy of particle

$$V^{(1)} = \frac{P_0 e - P e_0}{e_0 e - P P_0}$$

$$P_0 \equiv \sqrt{e_0^2 - f}$$

$$V^{(3)} = \frac{L\sqrt{-f}}{rP}$$

$$V^{(3)} = V_{st}^{(3)} \frac{\sqrt{f}}{1 + vV^{(1)}_{st}} = V_{st}^{(3)} \frac{\sqrt{1 - v^2}}{1 + vV^{(1)}_{st}}$$

$$V^{(1)}_{st} \to -1$$

Big Lorentz boost compensates small angular velocity in static frame.
As a result, in Lemaitre frame component

$V^{(3)}$ is finite and nonzero

In a similar way, radial velocity can take any value.

All this is a bright manifestation of the known relativistic effect according to which a vector, not collinear to the direction of motion, rotates under a Lorentz transformation.

Vicinity of singularity

$$r \to 0, f \to \infty$$

This means that any initial differences in radial motion for different particles disappear near the singularity,
and the radial motion of any particle tends to the motion of the frame.

Pure radial case

$$V^{(1)} \approx \frac{1-\varepsilon}{\sqrt{|f|}} \to 0$$

Non-radial case

$$V^{(1)} \approx \frac{1}{\sqrt{|f|}} \to 0$$

Nonradial motion with nonzero L

The situation with angular velocity is opposite. If

$$L \neq 0 \qquad V^{(3)} \to \pm 1$$

Pure radial motion appears to be unstable—an arbitrary small deviation grows infinitely and results in an ultrarelativistic motion in an angular direction. If initially the directions of the vectors L are distributed randomly, the corresponding particles have mutual ultrarelativistic relative velocities near a singularity.

Horizon limit in general

If e and e_0 have the same sign

$$V^{(1)} \to V_H{}^{(1)} = \frac{e_0^2(1+\frac{L^2}{r_g^2}) - e^2}{e_0^2(1+\frac{L^2}{r_g^2}) + e^2}$$

$$V^{(3)} \to V_H{}^{(3)} = \frac{2ee_0L}{r_g[e^2 + e_0^2(1+\frac{L^2}{r_g^2})]}.$$

For different signs of e e_0

$$|V^{(1)}| \to 1$$

$$V^{(3)} \to 0$$

Near singularity (now with arbitrary e and e_0)

$$L \neq 0 \qquad V^{(3)} \to signL = \pm 1$$

$$V^{(1)} \approx \frac{e_0}{P_0} \approx \frac{e_0}{\sqrt{-f}} \to 0$$

pure radial motion appears to be unstable—an arbitrary small deviation grows infinitely and results in an ultrarelativistic motion in an angular direction. If initially the directions of the vectors *L* are distributed randomly, the corresponding particles have mutual ultrarelativistic relative velocities near a singularity.

If $\quad e_0 = 0$

$$V^{(1)} = -\frac{e}{P} \approx -e\frac{r}{\sqrt{-f}\,|L|} \to 0$$

$$L = 0$$

$$V^{(3)} = 0$$

$$V^{(1)} \approx \frac{e_0 - e}{\sqrt{-f}} \to 0$$

Classification of frames

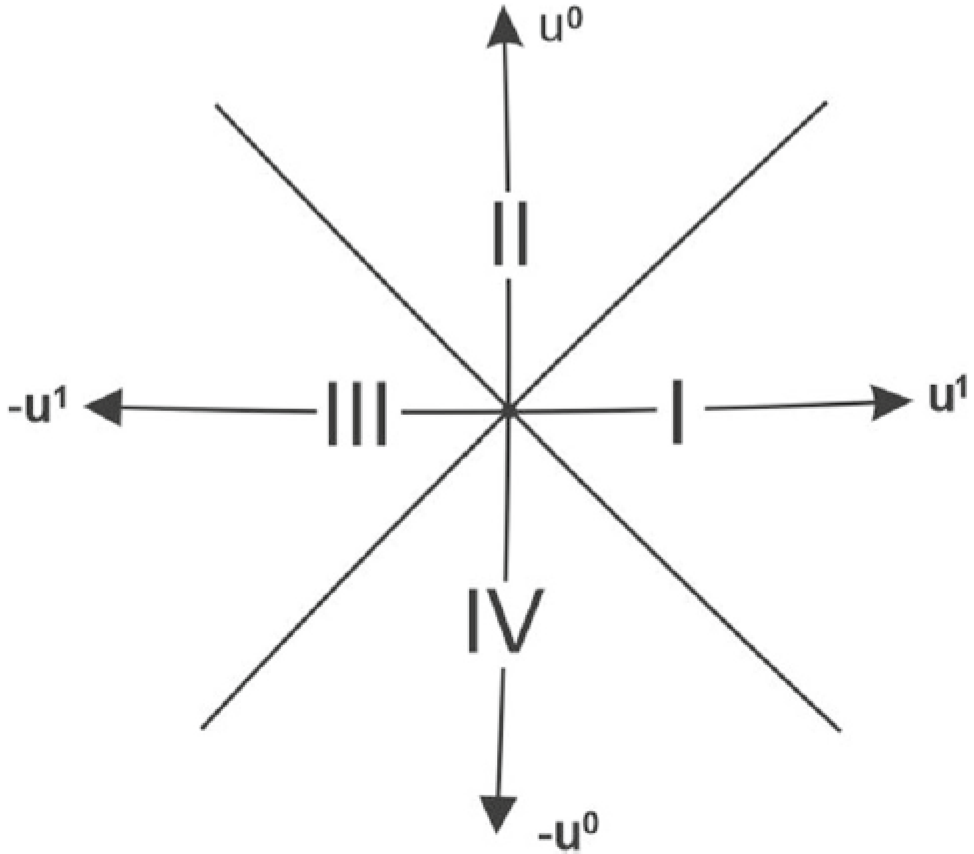


Fig. 1 The Kruskal diagram for the Schwarzschild metric

E – expanding, C – contracting, sign of e_0 indicated

Table 1 Four frames associated with the metric (25) and their arreas of existence in the Kruskal diagram

Frame	Regions covered
$(C, +)$	I, II
$(C, -)$	III, II
$(E, +)$	IV, I
$(E, -)$	IV, III

MOTION WITH ANGULAR MOMENTUM AND HORIZON ASYMPTOTICS

Frame / Particle	$e > 0,\ P > 0$	$e < 0,\ P > 0$	$e > 0,\ P < 0$	$e < 0,\ P < 0$
$e_0 > 0,\ P_0 > 0$	$V_H^{(1)};\ V_H^{(3)}$	$+1;\ 0$	$+1;\ 0$	–
$e_0 < 0,\ P_0 > 0$	$-1;\ 0$	$-V_H^{(1)};\ V_H^{(3)}$	–	$-1;\ 0$
$e_0 > 0,\ P_0 < 0$	$-1;\ 0$	–	$-V_H^{(1)};\ V_H^{(3)}$	$-1;\ 0$
$e_0 < 0,\ P_0 < 0$	–	$+1;\ 0$	$+1;\ 0$	$V_H^{(1)};\ V_H^{(3)}$

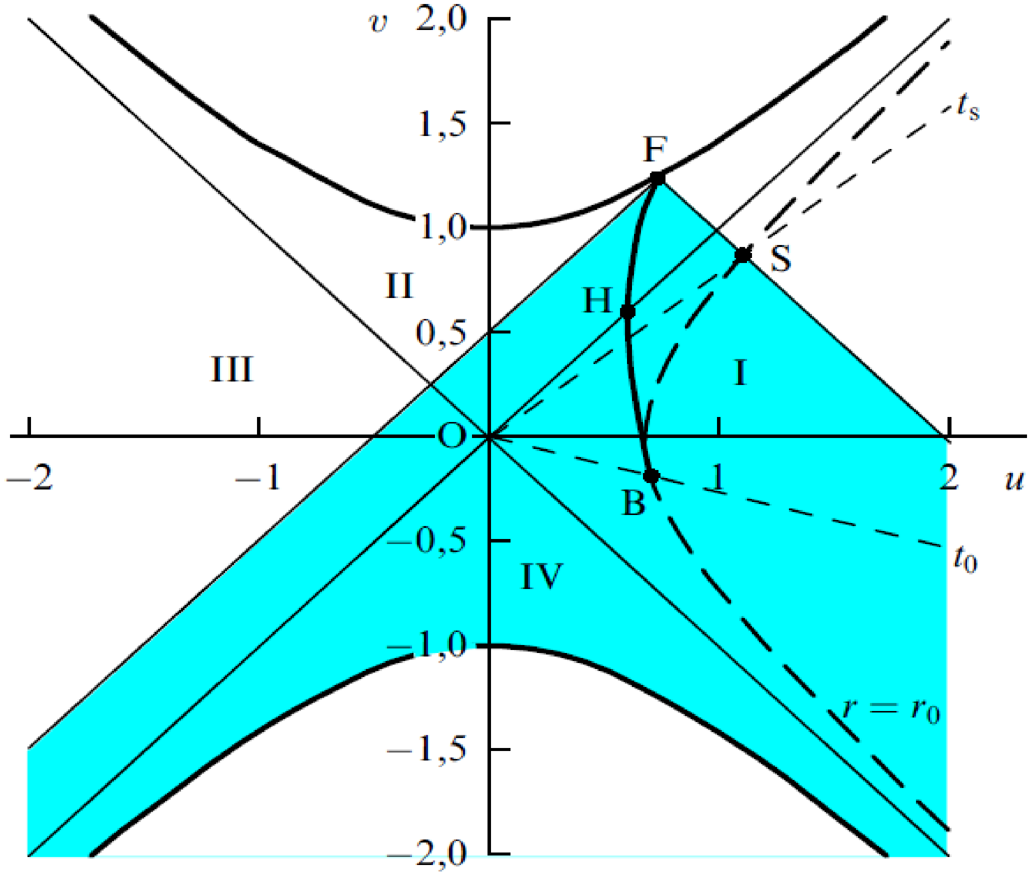
What Can a Falling Observer See?

Proper time finite, time t of remote observer infinite.
Finiteness of speed of light – only finite part of Universe is accessible to falling observer (Krasnikov 2008, Grib and Pavlov 2009)

Assumptions
1) Observer falls from right region R1
2) It moves along geodesics

But now an astrounaut should switch from E>0 to E=0.
Engine!

Grib and Pavlov 2009



Fall from $r_0 \le r_g$ to singularity

Trust at some point r_{0n} Switch from $\varepsilon = 1$ to $\varepsilon = 0$

$$\varepsilon = \frac{1 - v v_p}{\sqrt{1 - v^2{}_p}}$$

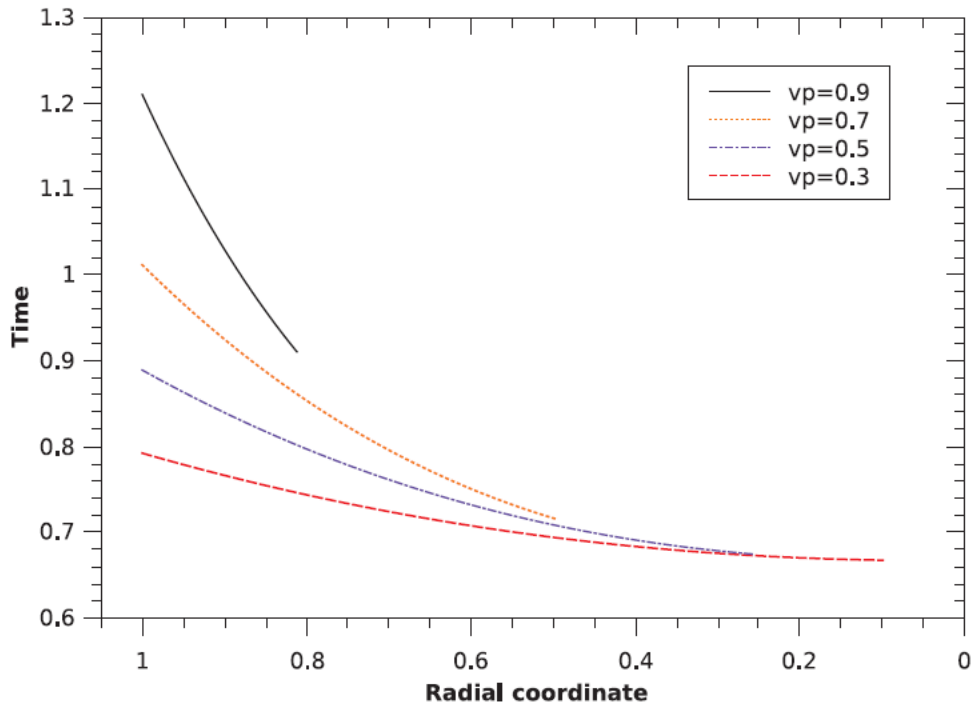
When $\varepsilon = 0$ $r = r_{cr} = r_g v^2{}_p$

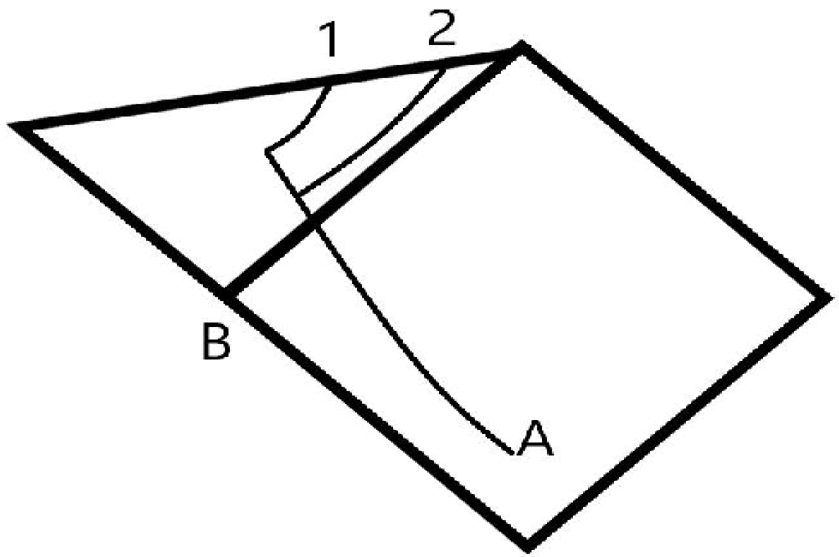
Geodesic before and after

$$\tau = -\left(\int_{r_0}^{r_{\text{on}}} \frac{\mathrm{d}r}{\sqrt{r_g / r}} + \int_{r_{\text{on}}}^{0} \frac{\mathrm{d}r}{\sqrt{r_g / r + (\varepsilon^2 - 1)}} \right).$$

Toporensky and Popov, Resonance 2023

Figure 1. Falling from the horizon. A radial coordinate at the horizontal axis corresponds to the point where the engine is turned on. The vertical axis gives the time of falling from the horizon to the center of the BH. Different curves correspond to different values of V_p.





Schematic picture of trajectories of an astronaut changing his path to the $E = 0$ trajectory

Ability of engine. Ability of austronaut to survive. Bends more to horizon, sees more and more from outer Universe. Engine is more powerful – austronaut sees bigger part of Universe.

More general formula for observer with any E between point 1 and singularity

$$\tilde{t} = \int_0^{r_1} \frac{dr}{v - v_p}$$

Under horizon

$$v > v_p$$

Therefore, the bigger the v_p, the bigger is the Lemaıtre time. So, the astronaut should use the remaining fuel — the fight against gravity makes sense! Ironically, not for the fighter — his proper time till singularity decreases while Lemaıtre time increases.

$$v_p = 1$$

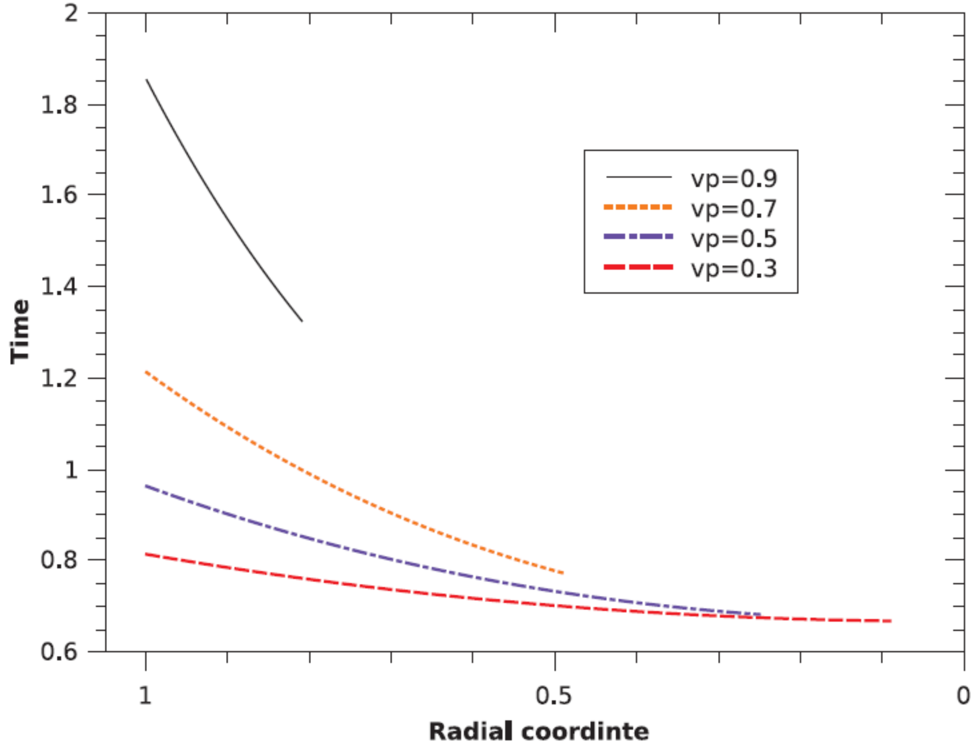
$$\tilde{t}_{max} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{r_g/r} - 1}.$$

$$\tilde{t}_{\max} > \tau_{\max}$$

$$\tau_{max} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{(r_g/r) - 1}}.$$

Toporensky and Popov 2023

Figure 2. Fall time from the horizon in the Lemaître frame. Different curves correspond to different values of V_p.



Two Strategies of an Astronaut

If one uses an engine near the horizon to make E=0, two goals at once: maximizing proper time till the singularity and maximizing the possible future of the universe seen during this fatal fall.
What happens if an engine is turned on deeply inside T region?
These two goals may require different strategies.

For example, suppose that the observer inside the
horizon found himself at a trajectory with $\varepsilon = 0$, but some fuel remains.
Is it
reasonable to use the fuel more? If we want to make the proper time before hitting
singularity as large as possible, the answer is obviously “no” — the trajectory with
$\varepsilon = 0$ is optimal. But what about the Lemaıtre time till the singularity?

If an astronaut understands
that he/she is actually on the trajectory with $\varepsilon < 0$ and wants to achieve the maximum possible proper time, it is necessary to decrease v_p in order to reach
$\varepsilon = 0$. On the contrary, such an astronaut should increase v_p as much as possible
to maximize the Lemaıtre time (allowing to see more future of the outer word).
In other words, a researcher inside the horizon should pay by the time of his own life for satisfying his curiosity!

Existential question: to live long but boring life or a short life but to learn something?

Better to combine both but under Schwarzschild horizon this is impossible

Contradictory strategies

	Goal 1	Goal 2		
E>0	Decrease E to E=0	Decrease E to E=0		
E=0	Do nothing	Increase v_p		
E<0	Decrease v_p	Increase v_p		

Goal 1: to make survival proper time bigger

Goal 2: to see the maximum from outer Universe

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Thank you!