Stable solutions in Horndeski theory

ArXiv: 2212.03285

by A. Shtennikova (MSU & INR RAS)

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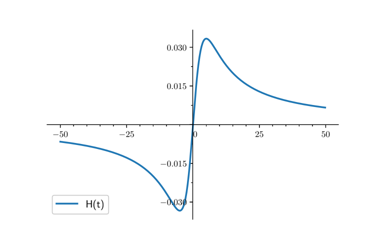
» **Some problems of Inflation**

- η-problem. In large field models of inflation, the inflaton has to traverse a distance in field space larger than the Planck mass M_{pl} in natural units. This has been argued to be problematic, since non-renormalizable quantum corrections to the field's action arise. In the absence of functional fine-tuning or additional symmetries, inflation would be spoiled;
- The presence of eternal inflation in almost all proposals has been argued to lead to a possible loss of predictability due to our inability to prescribe a unique measure: this is the so-called measure problem.
- Inflation does not provide a theory of initial conditions that would explain why the inflaton field starts out high in its potential.

A. Linde (2014), 1402.0526

» Alternative scenarios

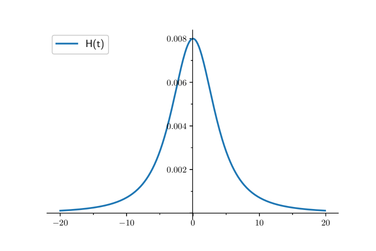
Bounce



Starts from contracting stage ⇒ bounce ⇒ expansion

M. Novello, S. E. Perez Bergliaffa (2008), arXiv: 0802.1634

Genesis



Starts from Minkowski, empty space, then energy density builds up, Universe starts to expand, expansion accelerates.

Creminelli, Nicolis, Trincherini (2010), arXiv: 1007.0027

Both can be viewed as alternatives to, or completion of inflation.

» Null energy condition (NEC)

$$T_{\mu\nu}n^{\mu}n^{\nu} \geq 0$$

for any null vector n^{μ}, such that $n_{\mu}n^{\mu} = 0$

* Quite robust
* Implies a number of properties. For example: Penrose theorem.

Penrose' 1965

In cosmology: if the NEC holds, and spatial curvature is negligible, there is initial singularity
$\Rightarrow$ No Bounce or Genesis.

* A combination of Einstein equations (spatially flat):

$$\frac{dH}{dt} = -4\pi G(\rho + p)$$

  $\rho = T_{00} =$ energy density; $T_{ij} = \delta_{ij} p =$ effective pressure.

* The Null Energy Condition:

$$T_{\mu v} n^{\mu} n^{\nu} \geq 0, n^{\mu} = (1,1,0,0) \Longrightarrow \rho + p \geq 0 \Longrightarrow dH/dt \leq 0,$$

  Hubble parameter was greater early on. **No bounce**

* Another side of the NEC: Covariant energy-momentum conservation:

$$\frac{d\rho}{dt} = -3H(\rho + p)$$

  NEC: energy density decreases during expansion, except for $p = -\rho$, cosmological constant. **No Genesis**

» Horndeski theory

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right],$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right],$$

where π is the scalar field, $X = g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$, $\pi_{,\mu} = \partial_\mu\pi$, $\pi_{;\mu\nu} = \nabla_\nu\nabla_\mu\pi$, $\Box\pi = g^{\mu\nu}\nabla_\nu\nabla_\mu\pi$, $G_{4X} = \partial G_4/\partial X$, etc.

» Perturbations

spatially flat FLRW background:

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right).$$

The decomposition of metric perturbations $h_{\mu\nu}$ into helicity components in the general case has the form

$$\begin{aligned}
h_{00} &= 2\Phi \\
h_{0i} &= -\partial_i \beta + Z_i^T, \\
h_{ij} &= -2\Psi\delta_{ij} - 2\partial_i\partial_j E - \left(\partial_i W_j^T + \partial_j W_i^T\right) + h_{ij},
\end{aligned}$$

the perturbation of scalar field $\delta\pi = \chi$.

The action for perturbations has the form:

$$S^{(2)} = \int \mathrm{d}t\ \mathrm{d}^3x a^3 \left[\frac{\mathcal{G}_T}{2}\left(\dot{h}_{ij}\right)^2 - \mathcal{F}_T \frac{\left(\overrightarrow{\nabla} h_{ij}\right)^2}{a^2} + \mathcal{G}_S\left(\dot{\zeta}\right)^2 - \mathcal{F}_S \frac{\left(\overrightarrow{\nabla}\zeta\right)^2}{a^2}\right].$$

$\mathcal{G}_T, \mathcal{F}_T, \mathcal{G}_S, \mathcal{F}_S$ - some functions on F, K, G_4, G_5 and their derivatives.

» No-go theorem

To avoid ghost and gradient instabilities one requires $\mathcal{G}_T > 0$, $\mathcal{G}_S > 0$ and $\mathcal{F}_T > 0$, $\mathcal{F}_S > 0$.
$\mathcal{F}_S$ has a structure:

$$\mathcal{F}_S = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}\xi - \mathcal{F}_T$$

$$\Rightarrow \frac{d}{dt}\xi = a \cdot (\mathcal{F}_S + \mathcal{F}_T) > 0,$$

The point is that

$$\xi = \frac{a\mathcal{G}_T^2}{2\theta},$$

is, therefore, a monotonously growing function, which means it must cross zero at some point, but we have $\mathcal{G}_T^2$ in the numerator of ξ. These two statements contradict each other.
Furthermore, at the point $\xi = 0$ $\theta \to \infty$ which means that the background fields H and π diverge, and that there is a singularity in the theory.

» Ways to avoid No-go theorem

1. Beyond Horndeski theory:

$$\mathcal{L} = \mathcal{L}_H + F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}$$
$$+F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'},$$

then $\xi = \frac{a\mathcal{G}_T(\mathcal{G}_T+D)}{2\theta}$ and one can construct stable non-singular solution.

S. Mironov, V. Rubakov, and V. Volkova (2018 - 2020), arXiv: 1807.08361, 1905.06249, 1910.07019.

2. Naive strong coupling:

$$\int_{-\infty}^{t} a(t)\xi dt = \int_{-\infty}^{t} a(t)\left[\mathcal{F}_T(t) + \mathcal{F}_S(t)\right] dt < \infty$$

This implies that $\mathcal{F}_T \to 0, \mathcal{F}_S \to 0$ as $t \to -\infty$; One also has $\mathcal{G}_T \to 0, \mathcal{G}_S \to 0$ as $t \to -\infty$. In this case, the coefficients in the quadratic action for perturbations about the classical solution tend to zero as $t \to -\infty$. It has been shown that, if we consider the next order of action for perturbations, the strong coupling can be avoided for a specific choice of the parameters of theory.

Y. Ageeva, P. Petrov and V. Rubakov (2020-2022), arXiv: 2009.05071, 2003.01202, 2104.13412

» $\theta = 0$

The quadratic action for the scalar perturbations has the form

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \Bigg(A_1\, \dot{\Psi}^2 + A_2\, \frac{(\overrightarrow{\nabla}\Psi)^2}{a^2} + A_3\, \Phi^2 + A_4\, \Phi \frac{\overrightarrow{\nabla}^2 \beta}{a^2} + A_5\, \dot{\Psi} \frac{\overrightarrow{\nabla}^2 \beta}{a^2}$$

$$+ A_6\, \Phi\dot{\Psi} + A_7\, \Phi\, \frac{\overrightarrow{\nabla}^2 \Psi}{a^2} + A_8\, \Phi \frac{\overrightarrow{\nabla}^2 \chi}{a^2} + A_9\, \dot{\chi} \frac{\overrightarrow{\nabla}^2 \beta}{a^2} + A_{10}\, \chi\ddot{\Psi} + A_{11}\, \Phi\dot{\chi}$$

$$+ A_{12}\, \chi \frac{\overrightarrow{\nabla}^2 \beta}{a^2} + A_{13}\, \chi \frac{\overrightarrow{\nabla}^2 \Psi}{a^2} + A_{14}\, \dot{\chi}^2 + A_{15}\, \frac{(\overrightarrow{\nabla}\chi)^2}{a^2} + A_{17}\, \Phi\chi$$

$$+ A_{18}\, \dot{\Psi}\chi + A_{19}\, \Psi\chi + A_{20}\, \chi^2 \Bigg)$$

where $A_4 = \theta$, $A_6 = 3\theta$, and $E = 0$ - partial gauge fix.

» Gauge invariant variables

This action is invariant with respect to small coordinate transformations:

$$x^\mu \to x^\mu - \xi^\mu,$$

where $\xi^\mu = \left(\xi_0, \xi_T^i + \delta^{ij}\partial_j \xi_S\right)^{\mathrm{T}}$. In which the fields change as:

$$\Phi \to \Phi + \dot{\xi}_0, \quad \beta \to \beta - \xi_0 + a^2 \dot{\xi}_S, \quad \chi \to \chi + \xi_0 \dot{\pi}, \quad \Psi \to \Psi + \xi_0 H, \quad E \to E - \xi_S.$$

The action can be rewritten in explicitly gauge-invariant form by introducing new variables (Bardeen variables):

$$\mathcal{X} = \chi + \dot{\pi}\left(\frac{\beta}{a^2} + \dot{E}\right),$$
$$\mathcal{Y} = \Psi + H\left(\frac{\beta}{a^2} + \dot{E}\right),$$
$$\mathcal{Z} = \Phi + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\beta}{a^2} + \dot{E}\right].$$

» Three variables action

In terms of these variables, the action will take the form

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \Bigg(A_1 \left(\dot{\mathcal{Y}}\right)^2 + A_2 \,\frac{(\vec{\nabla}\mathcal{Y})^2}{a^2} + A_3\, \mathcal{Z}^2 + A_6\, \mathcal{Z}\dot{\mathcal{Y}} + A_7\, \mathcal{Z}\,\frac{\vec{\nabla}^2\mathcal{Y}}{a^2}$$
$$+ A_8\, \mathcal{Z}\,\frac{\vec{\nabla}^2\mathcal{X}}{a^2} + A_{10}\, \mathcal{X}\ddot{\mathcal{Y}} + A_{11}\, \mathcal{Z}\dot{\mathcal{X}} + A_{13}\, \mathcal{X}\,\frac{\vec{\nabla}^2\mathcal{Y}}{a^2} + A_{14}\left(\dot{\mathcal{X}}\right)^2$$
$$+ A_{15}\,\frac{(\vec{\nabla}\mathcal{X})^2}{a^2} + A_{17}\mathcal{Z}\mathcal{X} + A_{18}\,\mathcal{X}\dot{\mathcal{Y}} + A_{20}\mathcal{X}^2\Bigg)$$

At this point it is clearly seen that the field Z is non-dynamic and we can derive a Z–constraint which has the following form:

$$\mathcal{Z} = \frac{1}{2A_3}\left(-A_7\frac{\vec{\nabla}^2\mathcal{Y}}{a^2} - A_8\frac{\vec{\nabla}^2\mathcal{X}}{a^2} + 3A_4\dot{\mathcal{Y}} - A_{11}\dot{\mathcal{X}} - A_{17}\mathcal{X}\right)$$

We used that $A_6 = -3A_4$.

» Only $A_4 = 0$ case

After integrating out Z, introducing

$$\zeta = \mathcal{Y} + \eta \mathcal{X}, \quad \eta = \frac{3A_{11}A_4 - 2A_{10}A_3}{4A_1A_3 - 9{A_4}^2},$$

and integrating out $\mathcal{X}$ variable, we get the following action:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left(A_2 \frac{\left(\overrightarrow{\nabla}\zeta\right)^2}{a^2} - \frac{1}{9} \frac{{A_1}^2}{A_3} \frac{\left(\overrightarrow{\nabla}^2\zeta\right)^2}{a^4} \right)$$

which means the absence of dynamics of the field ζ.

» Additional options

From the view of the Z–constraint,

$$\mathcal{Z} = \frac{1}{2A_3}\left(-A_7\frac{\overrightarrow{\nabla}^2\mathcal{Y}}{a^2} - A_8\frac{\overrightarrow{\nabla}^2\mathcal{X}}{a^2} + 3A_4\dot{\mathcal{Y}} - A_{11}\dot{\mathcal{X}} - A_{17}\mathcal{X}\right)$$

we can also distinguish the case $A_3 = 0$ as a singular point. By reason of the following ratios on the coefficients

$$A_3 = \frac{3}{2}A_4 H - \frac{1}{2}A_{11}\dot{\pi},$$

we have two options: $A_4 = 0, A_{11} = 0$ and $A_4 = 0, \dot{\pi} = 0$.

» $A_4 = 0, A_{11} = 0$

In this case, the Z–constraint gives us the condition:

$$\mathcal{X} = -\frac{A_7}{A_8}\mathcal{Y}$$

Which brings the action into the following form:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3\, m\mathcal{Y}^2$$

where

$$m = (\text{Some VERY big expression})$$

» $A_4 = 0, \dot{\pi} = 0$

In this case, the condition $A_4 = 0$ takes the form of:

$$G_4 H = 0$$

For $A_4 = 0$ it is also necessary to impose the condition $H = 0$. And the action takes the form:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left(\mathcal{G}_S \left(\dot{\mathcal{Y}}\right)^2 + m\mathcal{Y}^2 - \mathcal{F}_S \frac{\left(\overrightarrow{\nabla}\mathcal{Y}\right)^2}{a^2} \right)$$

Where $\mathcal{F}_S = \mathcal{G}_S$ The case of the Minkowski space in GR ($G_4 = \frac{1}{2}$) is a special case of this solution.

» Brief summary

<table>
<tr><td>$A_4 \neq 0$</td><td colspan="2">$c_\infty^2 = \mathcal{F}_S/\mathcal{G}_S$</td></tr>
<tr><td rowspan="2">$A_4 \equiv 0$</td><td>$\dot{\pi} \neq 0$</td><td>no dynamics in scalar sector</td></tr>
<tr><td>$\dot{\pi} = 0$</td><td>$c_\infty^2 = 1$</td></tr>
</table>

Thus, we obtained that $A_4 = 0$ everywhere, always leads to a stable solution in the scalar perturbation sector. In the case of non-trivial field π there are no dynamical scalar perturbations, and thus the stability condition does not arise at all, and in the case of a static background field π, we obtain a scalar perturbation with the sound speed squared $c_\infty^2 = 1$.

» Reconstruction of Lagrangian functions

Without loss of generality we choose the following form of the scalar field

$$\pi(t) = t,$$

so that $X = 1$. To reconstruct the theory which corresponds some solution we use the following ansatz for the Lagrangian functions

$$F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X,$$
$$K(\pi, X) = k_1(\pi) \cdot X,$$
$$G_4(\pi, X) = \frac{1}{2}.$$

We are interested to consider the case $G_4 = \text{const}$, which corresponds to GR.

Only the equations of motion and the condition $A_4 = 0$ remain as possible constraints:

$$f_0 = -\dot{H},$$
$$f_1 = -3H^2,$$
$$k_1 = H.$$

» Bouncing solution

Hubble parameter can be choosen in the following form for the case of the bounce:

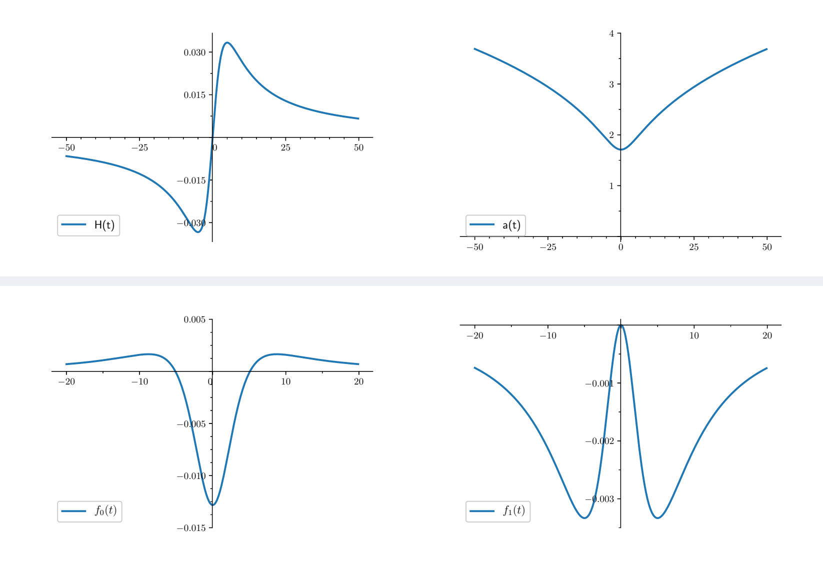
$$H(t) = \frac{t}{3\left(\tau^2 + t^2\right)},$$

so that

$$a(t) = \left(\tau^2 + t^2\right)^{\frac{1}{6}},$$

and the bounce occurs at $t = 0$. In what follows we take $\tau \gg 1$ to make this scale safely greater than Planck time. The parameter τ determines the duration of the bouncing stage.
Corresponding Lagrangian reads

$$\mathcal{L} = \frac{\pi^2 - \tau^2}{3\left(\tau^2 + \pi^2\right)^2} - \frac{\pi^2 X}{\left(\tau^2 + \pi^2\right)^2} + \frac{\pi X}{3\left(\tau^2 + \pi^2\right)}\Box\pi + \frac{1}{2}R.$$



Hubble parameter $H(t)$, scale factor $a(t)$ and the Lagrangian functions $f_0(t)$, $f_1(t)$ of the bouncing scenario with parameter $\tau = 25$ (recall that $k_1(t) = H(t)$).

» Genesis

Genesis case corresponds to the Hubble parameter with the following asymptotics on $t \to -\infty$:

$$H(t) \propto \frac{1}{(-t)^3}.$$

We consider full evolution which corresponds to a genesis start of the universe with subsequent slowing down to Minkowski space in the end. We choose

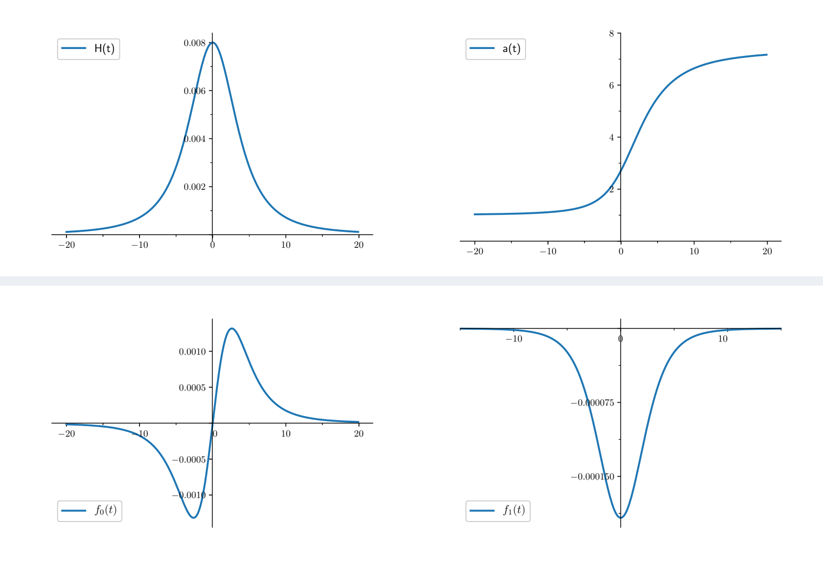
$$H(t) = \alpha \frac{\tau^2}{(t^2 + \tau^2)^{3/2}},$$

where α is an arbitary parameter which is responsible for the ratio of scale factors at + and $-\infty$. Then the scale factor is

$$a(t) = \exp\left(\frac{\alpha t}{\sqrt{\tau^2 + t^2}} + \alpha\right),$$

which is the solution to the background equations of motion of the Lagrangian:

$$\mathcal{L} = \frac{3\alpha\tau^2\pi}{(\tau^2 + \pi^2)^{5/2}} - 3X\frac{\alpha^2\tau^4}{(\tau^2 + \pi^2)^3} + X\frac{\alpha\tau^2}{(\tau^2 + \pi^2)^{3/2}}\Box\pi + \frac{1}{2}R.$$



Hubble parameter $H(t)$, scale factor $a(t)$ and the Lagrangian functions $f_0(t)$, $f_1(t)$ of the bouncing scenario with parameter $\alpha = 1, \tau = 25$ (recall that $k_1(t) = H(t)$).

Thank you for your attention!