

Topologically Quantized Schwarzschild Black Hole

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Introduction

The Schwarzschild (S) metric can be rewritten as ($c = G = 1$)

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} \right)} + r^2 f^2(\theta) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1)$$

where

$$f(\theta) = \pm k \frac{\sin \Theta}{\sin \theta} \quad (2)$$

with

$$\tan \frac{\Theta}{2} = \alpha \left(\tan \frac{\theta}{2} \right)^{\pm k} \quad (3)$$

in which $k \in \mathbb{N}$ and $\alpha = \text{const.}$

Introduction

Substituting this gives

$$f(\theta) = \frac{2k(\sin\theta)^{k-1}}{(1-\cos\theta)^k + (1+\cos\theta)^k}. \quad (4)$$

In general, we consider two Riemannian manifolds \mathcal{M} and \mathcal{M}' described by the metrics

$$\mathcal{M} : ds^2 = g_{ab}dx^a dx^b, \quad (a, b : 1, 2, \dots, n), \quad (5)$$

$$\mathcal{M}' : ds'^2 = g'_{AB}dx^A dx^B, \quad (A, B : 1, 2, \dots, m), \quad (6)$$

and a map $f^A : \mathcal{M} \rightarrow \mathcal{M}'$, ($A = 1, 2, \dots, m$) defined by the energy functional

$$E(f^A) = \frac{1}{2} \int g'_{AB} \frac{\partial f^A}{\partial x^a} \frac{\partial f^B}{\partial x^b} g^{ab} \sqrt{g} d^n x, \quad (7)$$

in which g is the determinant of g_{ab} . We note that n and m in our study are 2.

Introduction

This expression is adopted as the action so that the variational principle $\delta E(f^A) = 0$, yields the field equations

$$\nabla^2 f^A + \Gamma'_{BC}{}^A \frac{\partial f^B}{\partial x^b} \frac{\partial f^C}{\partial x^c} g^{bc} = 0. \quad (8)$$

Here ∇^2 stands for the covariant Laplacian on \mathcal{M} and $\Gamma'_{BC}{}^A$ are the connection coefficients on \mathcal{M}' .

Introduction

We refer to [1-4] for the details of the $f^A : S^2 \rightarrow S^2$ HM. The topic of HMs was considered long ago as an alternative formulation of physical theories [5,6] which will not be our aim in this Letter. Well-known classes of Einstein equations follow from (8) upon the appropriate choice of \mathcal{M} and \mathcal{M}' metrics which aided in obtaining new solutions [7-9].

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The formalism

Our choice for coordinates on \mathcal{M}' and \mathcal{M} are $f^A = \{\Theta, \Phi\}$ and $x^a = \{\theta, \varphi\}$, respectively, so that the two metrics are

$$\mathcal{M} : ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \quad (9)$$

and

$$\mathcal{M}' : ds'^2 = d\Theta^2 + \sin^2 \Theta d\Phi^2. \quad (10)$$

The choice $\Phi(\varphi) = k\varphi$, with $k \in \mathbb{N}$ (our choice will be $k > 0$), yields from the HM prescription the results (2) and (3) as the solution of the Eq. (8). The integer k denotes the number of times S^2 is wrapped in the map and if k is not an integer, (1) does not solve the vacuum Einstein equations. For $k = 1$, with the (+) sign, (3) reduces to the identity map, which gives the standard S-metric.

The formalism

We consider $k \geq 2$, so that a quantum number $n = \Delta k = k_2 - k_1 = \pm 1, \pm 2, \pm 3, \dots$, can be introduced. We note that with any $k \geq 1$ ($k \in \mathbb{N}$) the metric (1) is a vacuum solution of Einstein's equations. In other words, each k represents a Schwarzschild black hole. Therefore going from one solution to another, changes k for instance from k_1 to k_2 , such that $n = \Delta k$ represents this transition. Under the transform of S^2 the new Killing vectors (or angular momentum operators) take the form

$$L_x = \frac{1}{k} \cot \Theta \sin(k\varphi) \frac{\partial}{\partial \varphi} - \frac{1}{f(\theta)} \cos(k\varphi) \frac{\partial}{\partial \theta}, \quad (11)$$

$$L_y = \frac{1}{k} \cot \Theta \cos(k\varphi) \frac{\partial}{\partial \varphi} - \frac{1}{f(\theta)} \sin(k\varphi) \frac{\partial}{\partial \theta} \quad (12)$$

and

$$L_z = \frac{1}{k} \frac{\partial}{\partial \varphi} \quad (13)$$

which satisfy the original algebra $[L_i, L_j] = \epsilon_{ijk} L_k$, upon substitutions for Θ and $f(\theta)$ from Eqs. (3) and (4) above.

The formalism

Since (1) is a vacuum solution we need to add an energy term to match with any possible contribution as a quantum correction. This is done by considering a Reissner-Nordström (RN)-like term where an appropriate coefficient replaces the charge term (4). Upon this consideration, a possible choice for a topologically quantum-corrected S-metric takes the form

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{\left(1 - \frac{1}{k}\right) \ell_h^2}{r^2} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{\left(1 - \frac{1}{k}\right) \ell_h^2}{r^2} \right)} + r^2 f^2(\theta) d\Omega^2 \quad (14)$$

where the minimum length ℓ_h for HMs is added for dimensional reasons. At this point we leave it open whether ℓ_h is to be identified with the Planck length $\ell_P = \left(\frac{\hbar G}{c^3}\right)^{1/2}$, in which G is Newton's gravitational constant c is the speed of light and \hbar is the Planck constant.

The formalism

The surface area \mathcal{A} of the wrapped S^2 is given by

$$\mathcal{A} = 4\pi k r^2 \quad (15)$$

where for the area of the event horizon we take $r = r_h$. The energy of the map from (7) gives $E(f^A) = 4\pi k$, however, fixing this so that $k = 1$ corresponds to S-vacuum with the right physical dimensionality compels us to choose the energy quantum as

$$E = 4\pi (k - 1) \ell_h. \quad (16)$$

Let us note that (14) has the vacuum limit in $M = 0$ which consists entirely of quantum effects with frequency $\omega_n = \frac{4\pi |n| \ell_h}{\hbar}$.

The formalism

Here, the energy-momentum tensor is given by

$$T_{\mu}^{\nu} = \text{diag}(-\rho, p_r, p_{\theta}, p_{\varphi}) = \frac{(1 - \frac{1}{k}) \ell_h^2}{r^4} \text{diag}(-1, -1, 1, 1) \quad (17)$$

in which $\rho = \frac{(1 - \frac{1}{k}) \ell_h^2}{r^4}$ is the energy density such that the total conserved energy of the spacetime

$$E = \int \sqrt{-g} \rho d^3x = 2\pi \left(1 - \frac{1}{k}\right) \ell_h^2 \int_0^{\pi} f(\theta)^2 \sin \theta d\theta \int_{\ell_h}^{\infty} \frac{dr}{r^2} = 4\pi (k - 1) \ell_h. \quad (18)$$

The formalism

Let us clarify that $g = \det g_{\mu\nu} = -f^4 r^4 \sin^2 \theta$ in (18) is the determinant of the 3+1-dimensional spacetime, while $g = \det g_{ab} = f^4 r^4 \sin^2 \theta$ in Eq. (7) is the determinant of the angular part of the 3+1-dimensional spacetime. Furthermore, the Kretschmann scalar of (14) is given by

$$\mathcal{K} = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = \frac{48M^2 r^2 - 96Mr \left(1 - \frac{1}{k}\right) \ell_h^2 + 56 \left(\left(1 - \frac{1}{k}\right) \ell_h^2 \right)^2}{r^8}, \quad (19)$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of the 3+1-dimensional spacetime. The pressure components are also evaluated under the quantum assumption that $r \geq \ell_h$. In this manner both \mathcal{A} and E are quantized.

The formalism

The event horizon ($r_h = r_+$) and the inner horizon (r_- , which is absent in the classical S-metric) are

$$r_+ \simeq M \left(2 - \frac{(1 - \frac{1}{k}) \ell_h^2}{2M^2} \right) \quad (20)$$

$$r_- \simeq \frac{(1 - \frac{1}{k}) \ell_h^2}{2M}, \quad (21)$$

and in the extremal case, we have

$$r_+ = r_- = M = \sqrt{1 - \frac{1}{k}} \ell_h \quad (22)$$

which is valid only for microscopical black holes.

The formalism

At this level, the mass automatically becomes quantized by the quantum of length ℓ_h . It is observed from (18) that the quantum correction to the mass/energy is

$$\Delta M = 4\pi n \ell_h, \quad (n = \pm 1, \pm 2, \dots) \quad (23)$$

which will give the emission/absorption frequency as $\omega_n = \frac{4\pi|n|\ell_h}{\hbar}$. The area change $\Delta \mathcal{A}$ gets contributions from both $\Delta k = n$ and from ΔM , which sums up to give

$$\Delta \mathcal{A} \simeq 16\pi M^2 n \left\{ 1 + \frac{8\pi k}{M} \ell_h - \frac{(1 - \frac{1}{k}) \ell_h^2}{2M^2} + \mathcal{O}(\ell_h^3) \right\}. \quad (24)$$

In the bracket, the first term corresponds to the classical area change, whereas the second and third terms correspond to the semiclassical and quantum corrections, respectively. The quantum correction to the area is $-8\pi n (1 - \frac{1}{k}) \ell_h^2$, which is positive or negative depending on the absorption or emission of gravitational radiation.

The formalism

The quantized Hawking temperature and the entropy of the black hole (14) are given by

$$T_H = \left(-\frac{g'_{00}}{4\pi\sqrt{-g_{00}g_{11}}} \right)_{r=r_+} = \frac{\chi}{2\pi M(1+\chi)^2}, \quad (25)$$

and

$$S = \frac{\mathcal{A}(r_+)}{4} = k\pi M^2(1+\chi)^2, \quad (26)$$

respectively, where

$$\chi = \sqrt{1 - \frac{(1 - \frac{1}{k})\ell_h^2}{M^2}}. \quad (27)$$

Using (25) and (26) we calculate the heat capacity of the black hole for fixed k which is given by

$$C_k = \left(T_H \frac{\partial S}{\partial T_H} \right)_k = \frac{4\pi km^2 \chi (1+\chi)^2}{2(1-2\chi)}. \quad (28)$$

The formalism

At this stage, a simple application will be in order. The circular, radial geodesics for $\theta = \text{const.}$, $\varphi = \text{const.}$, are described by the reduced Lagrangian

$$\mathcal{L} = -\frac{1}{2}\psi\dot{t}^2 + \frac{1}{2\psi}\dot{r}^2 \quad (29)$$

where

$$\psi = 1 - \frac{2M}{r} + \frac{(1 - \frac{1}{k})\ell_h^2}{r^2} \quad (30)$$

and a dot stands for $\frac{d}{ds}$. This admits the dimensionless energy constant E_0 so that for $r = r_0 = \text{const.}$ we have

$$r_0 = \frac{M}{E_0^2 - 1} \left[\sqrt{1 + \frac{(E_0^2 - 1)(1 - \frac{1}{k})\ell_h^2}{M^2}} - 1 \right] \quad (31)$$

exactly, and

$$r_0 \simeq \frac{(1 - \frac{1}{k})\ell_h^2}{2M} \left[1 - \frac{(E_0^2 - 1)(1 - \frac{1}{k})}{4} \left(\frac{\ell_h^2}{M^2} \right) + \mathcal{O} \left(\left(\frac{\ell_h^2}{M^2} \right)^2 \right) \right] \quad (32)$$

in expansion.

The formalism

This defines the possible circular radii of a *quantum Schwarzschild atom* with mass M which collapses to $r_0 = 0$, for $k = 1$ to create the classical singularity. To the order $\frac{\ell_h^2}{M}$, the radius (32) takes the orbital values

$$r_0 \simeq \frac{\ell_h^2}{M} \frac{(1 - \frac{1}{k})}{2} = \frac{\ell_h^2}{M} \begin{cases} 1/4, & k = 2 \\ 1/3, & k = 3 \\ 3/8, & k = 4 \\ \dots & \dots \\ 1/2, & k \rightarrow \infty \end{cases} \quad (33)$$

and there is no need to state that as in the atomic model all states need not be filled in such a quantized Schwarzschild model.

The formalism

The inclusion of angular momentum parameter ℓ can be considered in the simplest form in the equilateral plane $\theta = \frac{\pi}{2}$ where the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\psi(r)\dot{t}^2 + \frac{1}{2\psi(r)}\dot{r}^2 + \frac{1}{2}k^2r^2\dot{\varphi}^2 \quad (34)$$

which still bears the imprints of the integer k . In terms of the constants $E = \psi(r)\dot{t}$ and $\ell = k^2r^2\dot{\varphi}$, and by introducing $u = \frac{1}{r}$, the quantized Kepler orbits reduce to the form

$$\frac{k^2}{\ell}(\varphi - \varphi_0) = \int^u \frac{du}{\sqrt{E^2 - (1 - 2Mu + (1 - \frac{1}{k})\ell_h^2 u^2)(1 + \frac{\ell^2}{k^2}u^2)}} \quad (35)$$

to be fixed in terms of the elliptic functions.

The formalism

Our formalism can be extended to other metrics that involve S^2 sector. Inserting the same substitutions as (2) and (3) into the Kerr metric with expected complications provide a similar quantization. For the RN case, it is similar to the S-metric that the charge term in the metric will be shifted according to $\frac{Q^2}{r^2} \rightarrow \frac{Q^2 + (1 - \frac{1}{k})\ell_h^2}{r^2}$, in which the charge Q will compete with the quantum of length. In the extremal case, we may argue about the quanta of charge, as in the case of mass. Going to higher dimensional maps such as $f : S^m \rightarrow S^n$, faces technical problems which may be overcome by resorting to the special Hopf polynomial type maps (5).

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Conclusion

- 1 Different from other researchers, we don't define the areal quantization but derive it by appealing to the geometric HMs of $S^2 \rightarrow S^2$.
- 2 Area and energy are quantized naturally in terms of the topological number $k \in \mathbb{N}$ which measures at the same time the degree of the map.
- 3 The topological degree of the HMs $k \in \mathbb{N}$ represents the quantum integer for a quantized S-black hole.
- 4 For non-integer k the vacuum equations are not satisfied and then fixing the problem with a suitable source may be interesting from the topological point of view.

Conclusion

- ⑤ The areal change $\Delta\mathcal{A}$ consists of classical, semiclassical, and quantal terms in reducing powers. Since $r \geq \ell_h$, the fundamental length, and $k \geq 2$ the $r = 0$ singularity of the S-metric is removed whereas the correspondence principle is provided by $k = 1$.
- ⑤ Although we left the choice of ℓ_h open, for the unity of quantum in the universe $\ell_h = \ell_P$ may be a reasonable choice. Otherwise, assuming that gravity remains non-cooperative with other fields, ℓ_h becomes a candidate for a new fundamental length in our universe.
- ⑤ As a next step, we may employ the local isometry between black holes and colliding waves to study the wave structure of the quantized Schwarzschild geometry.

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Thank you

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