Geometric ways of modifying gravity

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On top of the usual curvature, one can consider two other geometric quantities related to the spacetime connection:

$$\text{torsion} \qquad T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$$

and

$$\text{nonmetricity} \qquad Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu}.$$

Then it is easy to see that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu} \right) + \frac{1}{2} \left(T^{\alpha}{}_{\mu\nu} + T_{\nu}{}^{\alpha}{}_{\mu} + T_{\mu}{}^{\alpha}{}_{\nu} \right) - \frac{1}{2} \left(Q_{\mu\nu}{}^{\alpha} + Q_{\nu\mu}{}^{\alpha} - Q^{\alpha}{}_{\mu\nu} \right).$$

Gravity is very successfully described by the General Relativity theory of Albert Einstein. It is one of the best and most beautiful theories we have. Still, we are stubbornly trying to modify it.

There are mysteries in cosmology. What are the Dark Sectors? Was there inflation, and if yes then how? And if the problems such as H_0 tension are real, what are we making out of that?

On top of that, there are singularities, inherent and unavoidable. They are mostly hidden whenever one can imagine. But don't we want to have a better understanding of what is going on?

And let alone the puzzle of quantum gravity, together with our pathological belief in the mathematically horrendous quantum field theory approach.

And the amazing news we get is that it is extremely difficult to meaningfully modify the theory of General Relativity.

Simple models such as $f(R)$ are almost nothing new, and can be reformulated as an extra universal force mediated by a scalar field on top of the usual gravity. Deeper attempts at modifying it require exquisite care to not encounter with ghosts, or other bad instabilities, or total lack of well-posedness, or no reasonable cosmology available, or.... you name it!

And having the miserable lack of an undoubtful success, it makes all the good sense to try whatever crazy new geometry one can think of. And let it lead us to a better understanding.

One possible alternative approach is to describe gravity in terms of different geometry. The teleparallel framework works in terms of torsion instead of curvature.

In the orthonormal-tetrad-based description of gravity, one can naturally have torsionful connections without curvature or non-metricity by

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{A}\partial_{\mu}e^{A}_{\nu}.$$

Note the zero spin connection! (pure tetrad approach)
At least locally, every connection of this sort can be written like this, for some particular tetrad.

If we go beyond TEGR, or just reproducing GR, this framework is about more than just a metric. In general, different tetrads for the same metric are physically different objects.

I do not agree with the common opinion that it is necessary to have a locally Lorentz covariant description of teleparallel gravity, nor with another frequent opinion that such a description is severely problematic.

The zero-spin-connection tetrad has a clear geometric meaning: it is a covariantly constant basis of vector fields.

I would like to stress that, due to its very geometric meaning,

the teleparallel connection should not be invariant under local transformations of its defining tetrad.

However, there is no problem of rewriting the whole story in terms of some another tetrad. Moreover, it can sometimes be very convenient to do so.

To fix the notations, recall that the quest for TEGR action can start from observing that a metric-compatible connection $\Gamma^{\alpha}_{\mu\nu}$ with torsion differs from the Levi-Civita one $\overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}$ by a contortion tensor:

$$\Gamma^{\alpha}_{\mu\nu} = \overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}(g) + K^{\alpha}{}_{\mu\nu}$$

which is defined in terms of the torsion tensor $T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ as

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\alpha\mu\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right).$$

It is antisymmetric in the lateral indices because I ascribe the left lower index of a connection coefficient to the derivative, e.g. $\nabla_{\mu} T^{\nu} \equiv \partial_{\mu} T^{\nu} + \Gamma^{\nu}_{\mu\alpha} T^{\alpha}$.

The curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}$$

for the two different connections obviously has a quadratic in K expression in the difference. Then making necessary contractions, such as $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$, we can come to

$$0 = R = \overset{(0)}{R} + \mathbb{T} + 2 \overset{(0)}{\bigtriangledown}_{\mu} T^{\mu}.$$

Here $T_{\mu} \equiv T^{\alpha}{}_{\mu\alpha}$ is the torsion vector while the torsion scalar

$$\mathbb{T} \equiv \frac{1}{2} S_{\alpha\mu\nu} T^{\alpha\mu\nu}$$

is given in terms of the superpotential

$$S_{\alpha\mu\nu} \equiv K_{\mu\alpha\nu} + g_{\alpha\mu} T_{\nu} - g_{\alpha\nu} T_{\mu}.$$

Due to the basic relation above, the Einstein-Hilbert action $-\int d^4x\sqrt{-g}\overset{(0)}{R}$ is equivalent to the TEGR one, $\int d^4x\|e\|\mathbb{T}$. They are the same, up to the surface term $\mathbb{B} \equiv 2\overset{(0)}{\bigtriangledown}_\mu T^\mu$.

Of course, this equivalence disappears when we go to modified gravity, for example the $f(T)$ gravity:

$$S = \int f(\mathbb{T}) \cdot \|e\| d^4x.$$

Actually, the work of varying this action can be simplified a lot by using this observation.

But many problems await us!
The pesky strong coupling issues...

After some little exercise, the equation of motion can be written as

$$f' \overset{(0)}{G}_{\mu\nu} + \frac{1}{2}\left(f - f'\mathbb{T}\right) g_{\mu\nu} + f'' S_{\mu\nu\alpha}\partial^{\alpha}\mathbb{T} = \kappa \mathcal{T}_{\mu\nu}$$

with $\mathcal{T}_{\mu\nu}$ being the energy-momentum tensor of the matter.
This is a very convenient form of equations!
Addition of a flat spin connection does not make any change to it.

If $f'' \neq 0$, then the antisymmetric part of the equations takes the form of

$$(S_{\mu\nu\alpha} - S_{\nu\mu\alpha})\partial^{\alpha}\mathbb{T} = 0.$$

It can be thought of as related to Lorentzian degrees of freedom.

And we see that solutions with constant $\mathbb{T}$ are very special and do not go beyond the usual GR, unless we are to study perturbations around them.

The number of degrees of freedom is not very well known. And the main reason is a variable rank of the algebra of Poisson brackets of constraints.
But, what is for sure, is that there must be at least one extra mode.

Still, the trivial Minkowski $e^A_\mu = \delta^A_\mu$ is obviously in a strong coupling regime for a model with $f(0) = 0$ in vacuum. Indeed, then $\mathbb{T} \propto (\partial\delta e)^2$, and for the quadratic action we just take $f(\mathbb{T}) = f_0 + f_1\mathbb{T} + \mathcal{O}(\mathbb{T}^2)$ which means accidental restoration of the full Lorentz symmetry, and linearised GR.
Therefore, all the standard properties of gravitational waves are there. This absence of contradiction to experiments is highly problematic.

Moreover, the strong coupling issue is there also for the standard cosmology.

One can look for generalisations. For example, a model of $f(\mathbb{T}, \mathbb{B})$ type. Those go beyond one of the main initial motivations for $f(\mathbb{T})$ gravity, for they produce 4-th order equations of motion.

It is unclear whether they can avoid the Ostrogradski-type ghosts, unless in the case of $f(\overset{(0)}{R})$. However, what is clear is that they inherit all the troubles of $f(\mathbb{T})$ gravity. Indeed, they obviously can be rewritten as $f(\mathbb{T}, \overset{(0)}{R})$, with all the issues of rather chaotic remnant symmetries.

Then yet another option is non-metricity.
But can it be much better?

If we have only non-metricity and no torsion, $\Gamma^{\alpha}_{\mu\nu} = \mathring{\Gamma}^{\alpha}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}$ with the disformation being $L_{\alpha\mu\nu} = \frac{1}{2}\left(Q_{\alpha\mu\nu} - Q_{\mu\alpha\nu} - Q_{\nu\alpha\mu}\right)$ with the non-metricity $Q_{\alpha\mu\nu} = \partial_{\alpha} g_{\mu\nu}$.
Note the zero affine connection! (coincident gauge)
One easily relates the two curvature tensors and finds that

$$0 = \mathbb{R} = \mathring{\mathbb{R}} + \mathbb{Q} + \mathbb{B}$$

with

$$\mathbb{Q} = \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q_{\mu\alpha\nu} - \frac{1}{4} Q_{\mu} Q^{\mu} + \frac{1}{2} Q_{\mu} \tilde{Q}^{\mu},$$

$$\mathbb{B} = g^{\mu\nu} \mathring{\bigtriangledown}_{\alpha} L^{\alpha}{}_{\mu\nu} - \mathring{\bigtriangledown}^{\beta} L^{\alpha}{}_{\alpha\beta} = \mathring{\bigtriangledown}_{\alpha} \left(Q^{\alpha} - \tilde{Q}^{\alpha}\right)$$

where $Q_{\alpha} \equiv Q_{\alpha}{}^{\mu}{}_{\mu}$ and $\tilde{Q}_{\alpha} \equiv Q^{\mu}{}_{\mu\alpha}$.
Then all the game starts to resemble the metric teleparallel.

The diffeomorphisms become broken in $f(\mathbb{Q})$. But intuition tells me that it can hardly be in a stably broken way there.

There is still much to do.

There is a similar issue of fixing a gauge, pure tetrad in metric teleparallel and coincident gauge in symmetric teleparallel. From $\Gamma = 0$ by $x \longrightarrow \xi(x)$ one can get

$$\Gamma^{\alpha}_{\mu\nu} = \left[(\partial\xi)^{-1}\right]^{\alpha}_{\beta} \partial_{\mu}\partial_{\nu}\xi^{\beta}.$$

Note the second derivatives here!

Note also that the coincident-gauge STEGR is nothing but the non-covariant $\Gamma\Gamma$ action of Einstein.

Actually, the symmetric teleparallel geometry can be described in terms of a set of 1-forms that form a covariantly-constant basis

$$e^n_\mu \equiv \frac{\partial \xi^n}{\partial x^\mu},$$

or a (co-)tetrad with zero spin-connection, and the affine connection coefficients are

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_n \partial_\mu e^n_\nu.$$

Basically, the ζ^n are a set of coordinates in which the spacetime connection is zero.

A teleparallel geometry, i.e. zero curvature, means that there exists a basis of covariantly conserved vectors. It means

$$\nabla_{\mu} e^a_{\nu} = 0$$

in the sense of four independent 1-forms, or equivalently a soldering form which corresponds to zero spin connection.
In metric teleparallel, the usual approach is that this tetrad as a dynamical variable is absolutely free (for sure, apart from non-degeneracy), while the metric is defined as

$$g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu},$$

so that an arbitrary tetrad is orthonormal by definition.
In symmetric teleparallel, the tetrad is holonomic, i.e. it is a basis of coordinate vectors

$$e^a_{\mu} = \frac{\partial \xi^a}{\partial x^{\mu}},$$

while the metric is an independent variable.

All in all, I don't see it natural, to go for teleparallel formulations of gravity. The observed motion of matter is anyway of Riemannian nature.
(I don't go for "solving" the "problem of energy".)

Moreover, all the simplest teleparallel modifications do not seem to be viable, not even from purely theoretical perspective.

Notwithstanding all that, I think that studying them is very interesting and important, also for a better understanding of General Relativity itself.

Thank you!